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Massimiliano Berti

Nonlinear Oscillations of Hamiltonian PDEs

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Nonlinear Oscillations of Hamiltonian PDEs

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Preface

Many partial differential equations (PDEs) arising in physics can be seen as infinite-dimensional Hamiltonian systems

$$\partial_t u = J(\nabla_u H)(u,t)\,, \qquad u \in \mathcal{H}\,, \tag{0.1}$$

where the Hamiltonian function

$$H\colon \mathcal{H} \times \mathbf{R} \to \mathbf{R}$$

is defined on an *infinite*-dimensional Hilbert space $\mathcal{H}$ and J is a nondegenerate anti-symmetric operator.

The main examples of Hamiltonian PDEs are:

(a) the nonlinear wave equation

$$u_{tt} - \Delta u + f(x,u) = 0\,,$$

(b) the nonlinear Schrödinger equation

$$\mathrm{i}\, u_t - \Delta u + f(x,|u|^2)u = 0\,,$$

(c) the beam equation

$$u_{tt} + u_{xxxx} + f(x,u) = 0\,,$$

and the higher-dimensional membrane equation

$$u_{tt} + \Delta^2 u + f(x,u) = 0\,,$$

(d) the Euler equations of hydrodynamics describing the evolution of a perfect, incompressible, irrotational fluid under the action of gravity and surface tension, as well as its approximate models like the Korteweg–de Vries (KdV) equation, the Boussinesq equation, and the Kadomtsev–Petviashvili equation.
For example the celebrated KdV equation

$$u_t - uu_x + u_{xxx} = 0$$

was introduced to describe the motion of the free surface of a fluid in the small-amplitude, long-wave regime.

We start by describing the Hamiltonian structure of the 1-dimensional nonlinear wave equation with Dirichlet boundary conditions

$$\text{(NLW)} \qquad \begin{cases} u_{tt} - u_{xx} + f(t, x, u) = 0\,, \\ u(t, 0) = u(t, \pi) = 0\,, \end{cases}$$

or periodic boundary conditions $u(t, x + 2\pi) = u(t, x)$. In this book we shall consider both the autonomous nonlinear wave (f independent of time) and the nonautonomous nonlinear wave (forced case) equations.

Equation (NLW) can be seen as a second-order Lagrangian PDE

$$u_{tt} = -(\nabla_{L^2} W)(u, t) \tag{0.2}$$

defined on the infinite-dimensional "configuration space" $H_0^1(0, \pi)$, where

$$W(\cdot, t)\colon H_0^1(0, \pi) \to \mathbf{R}$$

is the "potential energy"

$$W(u, t) := \int_0^\pi \Big(\frac{u_x^2}{2} + F(t, x, u)\Big)\, \mathrm{d}x$$

and $(\partial_u F)(t, x, u) := f(t, x, u)$. Indeed, differentiating W and integrating by parts yields

$$\begin{aligned} \mathrm{d}_u W(u, t)\,[h] &= \int_0^\pi \Big(u_x h_x + f(t, x, u) h\Big)\, \mathrm{d}x \\ &= \int_0^\pi \Big(- u_{xx} + f(t, x, u)\Big) h \, \mathrm{d}x \end{aligned}$$

by the Dirichlet boundary conditions, so that the L^2-gradient of W with respect to u is

$$(\nabla_{L^2} W)(u, t) = -u_{xx} + f(t, x, u)\,,$$

and (NLW) can be written as (0.2).

The Lagrangian function of (NLW) is, as usual, the difference between the kinetic and the potential energy:

$$\mathcal{L}(u, u_t, t) := \int_0^\pi \frac{u_t^2}{2}\, \mathrm{d}x - W(u, t) = \int_0^\pi \Big(\frac{u_t^2}{2} - \frac{u_x^2}{2} - F(t, x, u)\Big)\, \mathrm{d}x\,.$$

The nonlinear wave equation (0.2) can be written also in Hamiltonian form

$$\begin{cases} u_t = p\,, \\ p_t = -(\nabla_{L^2} W)(u, t) = u_{xx} - f(t, x, u)\,, \end{cases}$$

and so in the form (0.1) with Hamiltonian function $H\colon H_0^1 \times L^2 \times \mathbf{R} \to \mathbf{R}$,

$$H(u,p,t) := \int_0^\pi \frac{p^2}{2}\,dx + W(u,t) = \int_0^\pi \Big(\frac{p^2}{2} + \frac{u_x^2}{2} + F(t,x,u)\Big)\,dx\,,$$

defined on the infinite-dimensional "phase space" $\mathcal{H} := H_0^1 \times L^2$, and the operator $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ represents a symplectic structure: the variables (u, p) are "Darboux coordinates."

Similarly we can deal with the nonlinear wave equation under periodic boundary conditions, choosing as configuration space $H^1(\mathbf{T})$, where $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$.

The Hamiltonian structure of the nonlinear Schrödinger equation (b), the beam and membrane equation (c), the KdV and the Euler equations (d) are presented in the appendix.

In this book we shall adopt the point of view of regarding the nonlinear wave equation (0.2) as an infinite-dimensional Hamiltonian system, focusing on the search for invariant sets for the flow.

The analysis of the principal structures of an infinite-dimensional phase space such as equilibria, periodic orbits, embedded invariant tori, center manifolds, as well as stable and unstable manifolds, is an essential change of paradigm in the study of hyperbolic equations with respect to the traditional pursuit of the initial value problem (mainly studied for dispersive equations).

Such a "dynamical systems philosophy" has led to many new results, known for finite-dimensional Hamiltonian systems, for Hamiltonian PDEs.

In the study of a complex dynamical system, the simplest invariant manifolds to look for are the equilibria and, next, the periodic orbits.

The relevance of periodic solutions for understanding the dynamics of a finite-dimensional Hamiltonian system was first highlighted by Poincaré.

In 1899 in "Les méthodes nouvelles de la Mécanique Céleste" Poincaré wrote (apropos periodic solutions in the three-body problem)

> *D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable.*

At first glance, knowledge of these does not seem really interesting because the periodic orbits form a zero-measure set in the phase space:

> *En effet, il y a une probabilité nulle pour que les conditions initiales du mouvement soient précisément celles qui correspondent à une solution périodique.*

But, as conjectured by Poincaré, their knowledge is not irrelevant:

> *voici un fait que je n'ai pu démontrer rigoureusement, mais qui me paraît pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13* [Hamilton's equations] *et une solution particulière quelconque de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut.* ([108], Tome 1, ch. III, a. 36).

This conjecture was the main motivation for the systematic study of periodic orbits that began with Poincaré and was then continued by Lyapunov, Birkhoff, Moser, Weinstein, and many others.

The first existence results, based on the Poincaré continuation method, regarded bifurcation of periodic orbits from elliptic (linearly stable) equilibria under nonresonance conditions on the eigenfrequencies of the small oscillations (Lyapunov center theorem [90]). These periodic solutions are the nonlinear continuations of the linear normal modes having periods close to the periods of the linearized equation.

Subsequently, more in the direction inspired by the conjecture of Poincaré, Birkhoff, and Lewis [32]–[34] proved that for "sufficiently nonlinear" Hamiltonian systems, there exist infinitely many periodic solutions with large minimal period in any neighborhood of an elliptic periodic orbit.

Great progress in understanding the very complex orbit structure of a Hamiltonian system was made by Kolmogorov [80], Arnold [8], and Moser [94] (KAM theory) in the 1960s: according to this theory, at least for sufficiently smooth nearly integrable systems, a set of positive measure of the phase space is filled by quasiperiodic solutions (KAM tori). The main difficulty in their existence proof is the presence of arbitrarily "small denominators" in their approximate expansion series.

From a dynamical point of view, these "small denominators" had already been highlighted by Poincaré as the main source of chaotic dynamics due to very complex resonance-type phenomena. Nowadays, this is the object of study of "Arnold diffusion" [9].

Returning to periodic orbits, in the 1970s, Weinstein [129], Moser [98], and Fadell–Rabinowitz [61] succeeded, with the aid of variational and topological methods in bifurcation theory, to extend the Lyapunov center theorem without nonresonance restrictions on the linear eigenfrequencies (resonant center theorems).

About the previous conjecture of Poincaré, some progress was achieved by Conley and Zehnder [46], who proved that the KAM tori lie in the closure of the set of the periodic solutions (having, it is true, very long minimal period). As a consequence, the closure of the periodic orbits has positive Lebesgue measure. Recent Birkhoff–Lewis-type results concerning lower-dimensional KAM tori—and applications to the three body problem—have been obtained in [23].

The conjecture of Poincaré was positively answered only in a generic sense by Pugh and Robinson [113] in the early 1980s: for a generic Hamiltonian (in the C^2 topology) the periodic orbits are dense on a compact and regular energy surface. On the other hand, for specific systems, the possibility of approximating any motion with periodic orbits remains an open problem.

Finally, the search for periodic solutions in Hamiltonian systems gave rise to many new ideas and techniques in critical point theory, especially after the breakthrough of Rabinowitz [117] and Weinstein [130] on the existence of a periodic solution on each compact and strictly convex energy hypersurface; see Ekeland [56], Hofer–Zehnder [72], and references therein.

Building on the experience gained from the qualitative study of finite-dimensional dynamical systems, the search for periodic solutions was regarded as a first step toward better understanding of the complicated flow evolution of Hamiltonian PDEs.

In this direction, Rabinowitz [118] and Brezis–Coron–Nirenberg [42] proved the existence of periodic solutions for nonlinear wave equations via minimax variational methods (see Theorem C.1). Interestingly, these proofs work only to find periodic orbits with a rational frequency, the reason being that other periods give rise to a "small denominators" problem type (Remark C.4).

Independently of these global results, in the late 1980s and early 1990s the local bifurcation theory of periodic and quasiperiodic solutions began to be extended to "nonresonant" Hamiltonian PDEs such as

$$u_{tt} - u_{xx} + a_1(x)u = a_2(x)u^2 + a_3(x)u^3 + \cdots \tag{0.3}$$

by Kuksin, Wayne, Craig, Bourgain, and Pöeschel, followed by many others.

Do there exist periodic and quasiperiodic solutions of (0.3) close to the infinite-dimensional "elliptic" equilibrium $u = 0$?

It turns out that in infinite dimension, the search for periodic solutions also exhibits a "small divisors difficulty," whereas in finite dimensions it arises only for finding quasiperiodic solutions.

The first existence proofs of small-amplitude periodic and quasiperiodic solutions were obtained by Kuksin [81] and Wayne [128]. These results were based on KAM theory and were initially limited to Dirichlet boundary conditions, because of the near coincidence of the linear eigenfrequencies under periodic spatial boundary conditions.

In an effort to avoid these restrictions Craig–Wayne [51] introduced the Lyapunov–Schmidt reduction method for Hamiltonian PDEs, extending the Lyapunov center theorem in the presence of periodic boundary conditions. Later, this method was generalized by Bourgain [35], [36] to construct quasiperiodic solutions. For subsequent developments of the KAM approach, see [109], [45], [84], [82], [59], [133], and references therein.

All the previous results concerned "nonresonant" PDEs, such as the nonlinear wave equation (0.3) with $a_1(x) \not\equiv 0$. The term $a_1(x)$ allows one to verify, as in the Lyapunov center theorem, suitable nonresonance conditions on the linear eigenfrequencies of the small oscillations, and the bifurcation equation is finite-dimensional.[1]

The aim of this book is to present recent bifurcation results of "nonlinear oscillations of Hamiltonian PDEs," especially for "completely resonant" nonlinear wave equations (0.3) with

$$a_1(x) \equiv 0$$

in which infinite-dimensional bifurcation phenomena appear jointly with small-divisor difficulties. We shall deal with both autonomous (time-independent) and

[1] For nonresonant PDEs, an extension of the Birkhoff–Lewis periodic orbit theory has also been obtained; see [14], [31].

forced (time-dependent) equations. In the autonomous case our results can be seen as infinite-dimensional analogues of the Weinstein–Moser and Fadell–Rabinowitz resonant center theorems. In [48] this case was pointed out as an important open question.

The plan of the book—which has a mainly didactical purpose—is as follows:

CHAPTER 1: We start with an overview in finite dimensions of the Lyapunov center theorem, the Weinstein–Moser and the Fadell–Rabinowitz resonant center theorems, concerning bifurcation of periodic solutions close to equilibria (nonlinear normal modes), respectively in the nonresonant and resonant cases.

CHAPTER 2: Next we consider infinite-dimensional Hamiltonian PDEs; we describe the "small divisors problem" and the presence of an infinite-dimensional bifurcation phenomenon dealing with "completely resonant" autonomous nonlinear wave equations. The infinite-dimensional bifurcation equation is solved via critical point theory, through a variant of the Ambrosetti–Rabinowitz mountain pass theorem. The exposition is self-contained (in the appendix we present the main ideas of critical point theory). This infinite-resonance variational analysis is the main new contribution of [24] (see also [25], [27], and [30]).

CHAPTER 3 contains a tutorial in Nash–Moser theory, which has been developed to overcome situations in which the standard implicit function theorem does not apply because the linearized operator is invertible only with a "loss of derivatives" (due, for example, to the "small divisors"). We present first a simplified form of this theorem (extracted from the final chapter of Zehnder in [103]) to highlight the main features of the method in an analytic setting. For completeness, we deal also with a simple differentiable Nash–Moser theorem.

CHAPTER 4: Later we prove a Nash–Moser-type theorem for completely resonant nonlinear wave equations, implying the existence of periodic solutions for sets of frequencies of asymptotically full measure. We emphasize the inversion of the linearized equation which is the most delicate step of any Nash–Moser implicit function theorem. The required spectral analysis is new (see [26]) and different from that of Craig–Wayne and Bourgain, allowing us to deal with nonlinearities with low regularity and without oddness assumptions.

CHAPTER 5: Finally, we present existence results for periodic solutions with rational frequency, in the forced case. The infinite-dimensional bifurcation equation (which now, unlike the autonomous case, is completely degenerate) is solved via purely variational methods (not of Nash–Moser type). This approach, developed as in [22], is based on constrained minimization, and it is related to elliptic regularity theory as in Rabinowitz [114].

APPENDIX: We present the Hamiltonian formulation of some PDEs including the Euler equation of hydrodynamics. We also discuss the basic notions in critical point theory that are used throughout the book. Finally, we prove, via minimax methods, Rabinowitz's result [118] about existence of one periodic solution of a superlinear nonlinear wave equation with rational frequency, following the proof of Brezis–Coron–Nirenberg [42].

This monograph is based on the material of a series of PhD courses the author delivered at the International School for Advanced Studies (SISSA) in recent years. Part of the results discussed here have appeared in joint work with Luca Biasco and Philippe Bolle. I wish to thank these friends warmly for many hours of enjoyable work together.

This text is far from being an exhaustive treatise on the theory of Hamiltonian partial differential equations, but is rather an introduction to the research in this fascinating and rapidly growing field.

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Napoli, November 2006 *Massimiliano Berti*

Contents

1

Finite Dimension

Let us consider a finite-dimensional dynamical system

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n, \tag{1.1}$$

where the vector field

$$f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$$

possesses an equilibrium at $x = 0$, i.e., $f(0) = 0$.

- QUESTION: do there exist periodic solutions of (1.1) close to $x = 0$?

  The first thing is to study the linearized system

$$\dot{x} = Ax, \qquad A := (D_x f)(0). \tag{1.2}$$

A necessary condition to find periodic solutions of the nonlinear system (1.1) close to the equilibrium $x = 0$ is the presence of *purely imaginary* eigenvalues of A. Indeed, if all the eigenvalues of A have nonzero real part (i.e., the matrix A is hyperbolic), then the Hartmann–Grobmann theorem (see, e.g., [68]) ensures a topological conjugacy between the linear and the nonlinear flows close to the equilibrium; see Figure 1.1. In this case, since the linear system (1.2) does not possess periodic solutions, neither does the nonlinear system (1.1).

Therefore, if A has the eigenvalues

$$\lambda_1, \dots, \lambda_n,$$

not necessarily distinct, we shall assume (recalling A is a real matrix) that

$$\lambda_1 = \mathrm{i}\omega, \qquad \lambda_2 = -\mathrm{i}\omega \tag{1.3}$$

is a pair of purely imaginary eigenvalues with complex eigenvectors

$$\xi = e_1 + \mathrm{i}e_2, \quad \overline{\xi} = e_1 - \mathrm{i}e_2, \quad e_1, e_2 \in \mathbf{R}^n.$$

Then $Ae_1 = -\omega e_2$, $Ae_2 = \omega e_1$ and the real plane

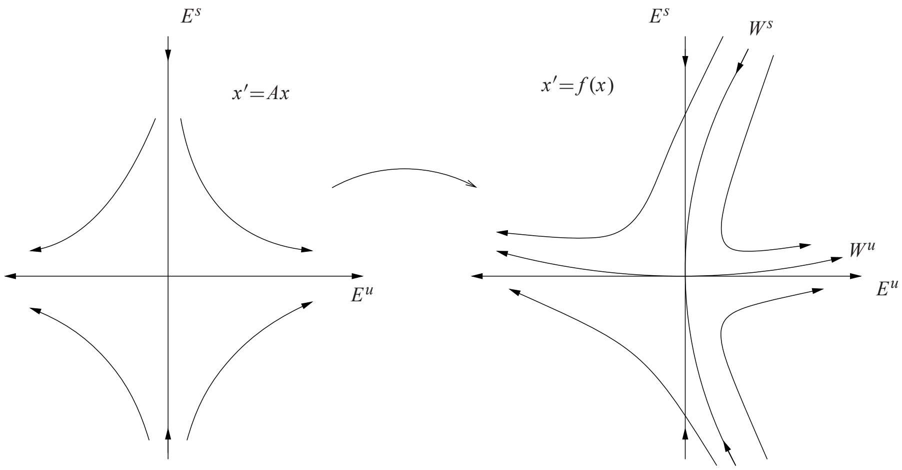

Fig. 1.1. The Hartmann–Grobmann theorem.

$$E := \mathrm{span}\{e_1, e_2\} \subset \mathbf{R}^n \tag{1.4}$$

is invariant for the linear dynamics of (1.2), and it is filled up by the $(2\pi/\omega)$-periodic solutions

$$x(t) = \mathrm{Re}\Big(c(e_1 + \mathrm{i}e_2)\mathrm{e}^{\mathrm{i}\omega t}\Big), \quad c \in \mathbf{C}.$$

See Figure 1.2. The quantity ω is called the frequency.

Remark 1.1. If $\omega = 0$, i.e., if A is singular, the plane E is filled by equilibria. In the sequel we shall give sufficient conditions to find periodic solutions of (1.1) in the case that $(D_x f)(0)$ is nonsingular.

Definition 1.2. The matrix A is called elliptic iff all its eigenvalues are purely imaginary. In this case the equilibrium $x = 0$ is called elliptic.

We could expect periodic solutions of (1.1) close to E. However, in general, periodic orbits need not exist in the nonlinear system, as the following simple example shows:

$$\begin{cases} \dot{x}_1 = -x_2 + (x_1^2 + x_2^2)x_1 \,, \\ \dot{x}_2 = x_1 + (x_1^2 + x_2^2)x_2 \,. \end{cases} \tag{1.5}$$

Here $(x_1, x_2) = (0, 0)$ is an equilibrium of (1.5) and the eigenvalues of the linearized system

$$\begin{cases} \dot{x}_1 = -x_2 \,, \\ \dot{x}_2 = x_1 \,, \end{cases}$$

are $\pm\mathrm{i}$. But for any solution $x(t) = (x_1(t), x_2(t))$ of (1.5),

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1^2(t) + x_2^2(t)) = 2(x_1^2(t) + x_2^2(t))^2 \,,$$

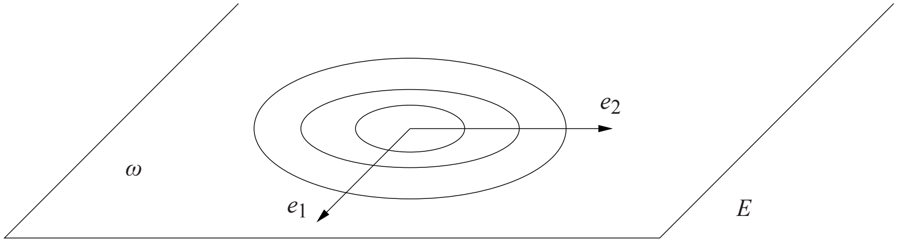

Fig. 1.2. The invariant plane E of the linear system (1.2) filled by $2\pi/\omega$-periodic solutions.

so that if a periodic solution $x(t)$ exists, whatever its period T is,

$$0 = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(x_1^2(t) + x_2^2(t))\,\mathrm{d}t = \int_0^T 2(x_1^2(t) + x_2^2(t))^2\,\mathrm{d}t\,,$$

and then $x(t) \equiv 0$.

1.1. Exercise: Make the phase portrait of (1.5).

The previous example shows that in order to find periodic solutions of (1.1), the class of the vector fields $f(x)$ *must* be restricted.

Two typical conditions are assumed:

(i) Systems with a prime integral (for example Hamiltonian systems);
(ii) Reversible systems.

In this book we shall consider only the class (i).

Definition 1.3. A smooth function $G\colon \mathbf{R}^n \to \mathbf{R}$ is a prime integral of (1.1) iff $G(x(t)) =$ constant along any solution $x(t)$ of (1.1). Equivalently,

$$\nabla G(x) \cdot f(x) = 0\,, \qquad \forall x \in \mathbf{R}^n\,. \tag{1.6}$$

In other words, every orbit evolves within the level set $\{G(x) = G(x(0))\}$ of the prime integral G; see Figure 1.3.

The main example of systems possessing a first integral is that of the autonomous Hamiltonian systems

$$\text{(HS)} \qquad \dot{x} = J\nabla H(x)\,, \qquad x \in \mathbf{R}^{2n}\,,$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathrm{Mat}(2n \times 2n)$$

is the standard symplectic matrix, I is the identity matrix in $\mathbf{R}^n$, and the function

$$H\colon \mathbf{R}^{2n} \to \mathbf{R}$$

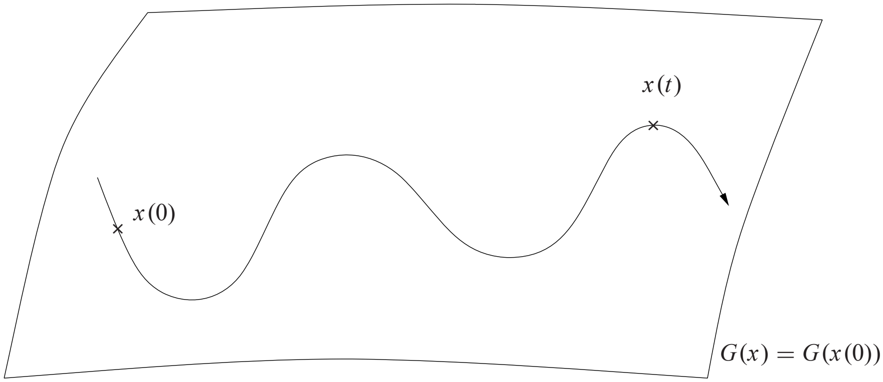

Fig. 1.3. Every orbit evolves within the level set $\{G(x) = G(x(0))\}$ of the prime integral G.

is called the Hamiltonian of (HS).

The Hamiltonian H itself is a prime integral of (HS) because

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = \nabla H(x(t)) \cdot J \nabla H(x(t)) = 0$$

by the antisymmetry of J, namely the transposed matrix $J^T = -J$.

Remark 1.4. Denoting the variables $x = (q, p) \in \mathbf{R}^{2n}$ by $q = (q_1, \ldots, q_n) \in \mathbf{R}^n$ and $p = (p_1, \ldots, p_n) \in \mathbf{R}^n$, the Hamiltonian system (HS) assumes the classical form

$$\begin{cases} \dot{q} = \partial_p H(q, p)\,, \\ \dot{p} = -\partial_q H(q, p)\,. \end{cases}$$

Considering the symplectic 2-form ω: $\mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}$ defined by

$$\omega := \mathrm{d}q \wedge \mathrm{d}p := \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i\,,$$

the Hamiltonian vector field $X_H(q, p) := (\partial_p H(q, p), -\partial_q H(q, p)) \in \mathbf{R}^{2n}$ is uniquely defined, by the nondegeneracy of ω, by

$$\mathrm{d}H(q, p)\,[h] = \omega(X_H(q, p),\, h\,) = X_H(q, p) \cdot Jh\,, \quad \forall h \in \mathbf{R}^{2n}\,.$$

1.1 The Lyapunov Center Theorem

We first analyze the form of a prime integral.

Lemma 1.5. *Suppose G is a C^2 prime integral of (1.1), $f(0) = 0$, $A := (D_x f)(0)$ possesses a pair of purely imaginary eigenvalues (cf. (1.3)) and* $\det A \neq 0$. *Then*

$(\nabla G)(0) = 0$ and there exists $\rho \in \mathbf{R}$ such that $(D^2G)_{|E}(0) = \rho I$, where E is defined in (1.4).

As a consequence, if $(D^2G)_{|E}(0)$ is nonsingular, then $(D^2G)_{|E}(0)$ is either positive or negative definite.

Proof. Differentiating (1.6), we obtain

$$(D^2G)(x)\xi \cdot f(x) + (\nabla G)(x) \cdot (D_x f)(x)\xi = 0\,, \qquad \forall \xi \in \mathbf{R}^n\,. \tag{1.7}$$

Therefore, since $f(0) = 0$, $(\nabla G)(0) \cdot A\xi = 0$, $\forall \xi \in \mathbf{R}^n$, and $(\nabla G)(0) = 0$ since A is nonsingular.

Differentiating (1.7) at $x = 0$, we obtain

$$2(D^2G)(0)\xi \cdot A\xi = 0\,, \qquad \forall \xi \in \mathbf{R}^n\,. \tag{1.8}$$

Restricting (1.8) to E, we obtain, $\forall \xi = \xi_1 e_1 + \xi_2 e_2$, $\xi_i \in \mathbf{R}$,

$$\begin{aligned} 0 &= (D^2G)(0)\xi \cdot A\xi \\ &= \omega\Big[-\xi_1^2 (D^2G)(0)e_1 \cdot e_2 + \xi_1\xi_2 \Big[(D^2G)(0)e_1 \cdot e_1 - (D^2G)(0)e_2 \cdot e_2\Big] \\ &\qquad + \xi_2^2 (D^2G)(0)e_1 \cdot e_2\Big]\,, \end{aligned}$$

i.e., $(D^2G)(0)e_1 \cdot e_2 = 0$, $(D^2G)(0)e_1 \cdot e_1 = (D^2G)(0)e_2 \cdot e_2 = \rho$. Therefore

$$(D^2G)(0)\xi \cdot \xi = \rho(\xi_1^2 + \xi_2^2)\,, \qquad \forall \xi \in E\,,$$

and if $(D^2G)_{|E}(0)$ is nonsingular, then $\rho \neq 0$ and $(D^2G)_{|E}(0)$ is either positive or negative definite. ■

We can now state the first continuation theorem of periodic orbits close to an equilibrium.

Theorem 1.6. (Lyapunov 1907 [90]) *Let $\lambda_1 = \mathrm{i}\omega$, $\lambda_2 = -\mathrm{i}\omega \neq 0$ be a pair of purely imaginary eigenvalues of the nonsingular matrix $A := (D_x f)(0)$ and let E be the corresponding subspace (1.4).*

If the other eigenvalues of A, λ_k, $k = 3, \ldots, n$, satisfy the "nonresonance condition"

$$\frac{\lambda_k}{\lambda_1} \notin \mathbf{Z} \tag{1.9}$$

and if G is a C^2 prime integral of (1.1) with $(D^2G)_{|E}(0) \not\equiv 0$, then for every small ε, there exists a unique $T(\varepsilon)$-periodic solution $p(\varepsilon, t)$ of (1.1) near E, on the level set[1] *$\{G(x) = G(0) + \varepsilon^2\}$, with period $T(\varepsilon)$ near $2\pi/\omega$.*

As $\varepsilon \to 0$ the solution $p(\varepsilon, t) \overset{L^\infty}{\to} 0$ and $T(\varepsilon) \to 2\pi/\omega$.

[1] By Lemma 1.5, $(D^2G)_{|E}(0)$ is either positive or negative definite; to fix notation we assume $(D^2G)_{|E}(0) > 0$. In the case $(D^2G)_{|E}(0) < 0$ the periodic solution $p(\varepsilon, t)$ lies in the level set $\{G(x) = G(0) - \varepsilon^2\}$.

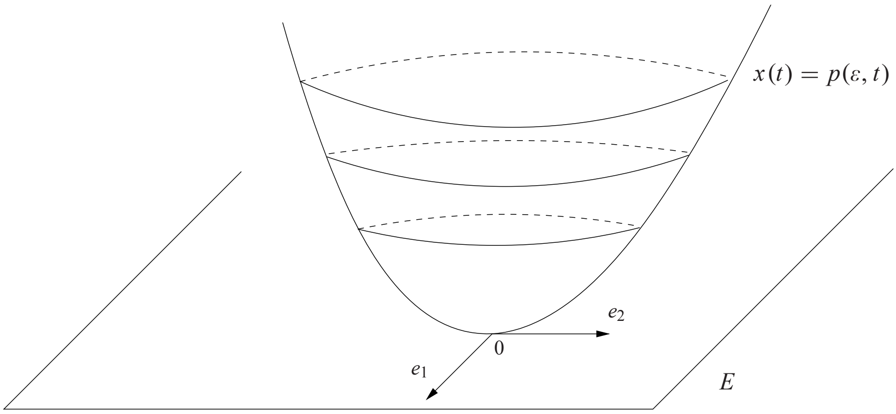

Fig. 1.4. The Lyapunov periodic solutions $p(t, \varepsilon)$ fill a 2-dimensional embedded disk that is tangent to the plane E at $x = 0$.

The Lyapunov periodic orbits $p(t, \varepsilon)$ of Theorem 1.6 are parametrized by the value $\{G(p(t, \varepsilon)) = G(0)+\varepsilon^2\}$ of the first integral G; other parametrizations are possible; see [6]. Note that in general, the Lyapunov periodic orbits are not parametrized by their frequency: it may happen that all their frequencies are the same, as, for example, in linear systems.

Remark 1.7. **(Regularity)** If $f \in C^{r+1}$, the Lyapunov periodic solutions fit together into a two-dimensional embedded invariant manifold that is C^r and tangent to the plane E at $x = 0$; see Figure 1.4. If f and G are analytic, then the embedding is analytic; see [100].

Remark 1.8. **(Multiplicity)** By Theorem 1.6, if there are s pairs of purely imaginary eigenvalues

$$\lambda_j = \pm \mathrm{i}\omega_j\,, \quad j = 1, 2, \ldots, s\,, \tag{1.10}$$

we obtain s distinct families of periodic solutions having periods close to $2\pi/\omega_j$ under the nonresonance conditions

$$\frac{\lambda_k}{\lambda_j} \notin \mathbf{Z} \quad \forall k \neq j\,, \quad j = 1, 2, \ldots, s \tag{1.11}$$

(and if $(D^2G)(0)$ is nonsingular on the corresponding eigenspaces).

Remark 1.9. **(Existence)** If there are s pairs of purely imaginary eigenvalues $\lambda_j = \pm\mathrm{i}\omega_j$ with $0 < \omega_1 \leq \cdots \leq \omega_{s-1} < \omega_s$ (namely λ_s is simple), then the Lyapunov center theorem ensures the persistence of the periodic orbits with the shortest period (highest frequency ω_s). Indeed, the nonresonance condition (1.11) is satisfied because

$$\frac{\lambda_k}{\lambda_s} = \frac{\mathrm{i}\omega_k}{\mathrm{i}\omega_s} < 1 \qquad \forall k = 1, \ldots, s-1$$

and $\lambda_j/(\mathrm{i}\omega_s) \notin \mathbf{Z}$ for the other eigenvalues $\lambda_j \notin \mathrm{i}\mathbf{R}$.

The other families need not exist, as the following example [101] shows:

$$\begin{cases} \dot{z}_1 = \mathrm{i}(jz_1 + \bar{z}_2^j)\,, \\ \dot{z}_2 = \mathrm{i}(-z_2 + j\bar{z}_1\bar{z}_2^{j-1})\,, \qquad j \in \{2, 3, \ldots\}\,, \end{cases} \tag{1.12}$$

where in complex notation, $z_k = q_k + \mathrm{i}p_k \in \mathbf{C}$. The function

$$G = j|z_1|^2 - |z_2|^2$$

is a prime integral of (1.12) with nonsingular $(D^2G)(0)$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}(z_1 z_2) = (|z_2|^2 + j^2|z_1|^2)|z_2|^{2j-2} \geq 0\,,$$

so that periodic orbits have $z_2 = 0$, and therefore

$$z_1 = c\,\mathrm{e}^{\mathrm{i}jt}\,,\; z_2 = 0\,,$$

gives the solutions of shortest period.

1.2. Exercise: Find explicitly the Lyapunov periodic solutions of the Hamiltonian system in $(\mathbf{R}^6, \sum_{i=1}^3 dq_i \wedge dp_i)$ with Hamiltonian

$$H = \frac{1}{2}\omega(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2) - \frac{1}{2}(q_3^2 + p_3^2) + (p_1^2 + q_1^2)(q_2 q_3 + p_2 p_3)$$

when the nonresonance condition $1 \neq l\omega$, $\forall l \in \mathbf{Z}$, holds.

We shall not prove the Lyapunov center theorem, which, thanks to the nonresonance condition (1.9) and the existence of the prime integral G, ultimately reduces to the implicit function theorem; see, e.g., [101] (as a particular case of the Poincaré continuation theorem) and [6] (as a particular case of the Hopf bifurcation theorem).

1.2 The Weinstein–Moser and Fadell–Rabinowitz Resonant Center Theorems

If the nonresonance condition (1.9) among the eigenvalues of A is violated (resonant case), no periodic solutions except the equilibrium point need exist.

An example due to Moser [99] is provided by the Hamiltonian system in $(\mathbf{R}^4, \sum_{i=1}^2 \mathrm{d}q_i \wedge \mathrm{d}p_i)$ generated by the Hamiltonian

$$H = \frac{q_1^2 + p_1^2}{2} - \frac{q_2^2 + p_2^2}{2} + \left(q_1^2 + p_1^2 + q_2^2 + p_2^2\right)(p_1 p_2 - q_1 q_2)\,. \tag{1.13}$$

Here $(q, p) = 0$ is an elliptic equilibrium and the eigenvalues of the linearized system are $\pm\mathrm{i}$, $\pm\mathrm{i}$ (not simple). But

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_1 p_2 + p_1 q_2) = 4(p_1 p_2 - q_1 q_2)^2 + \left(q_1^2 + p_1^2 + q_2^2 + p_2^2\right)^2,$$

so that the unique periodic solution is $q = p = 0$.

Remark 1.10. It may be useful, in order to understand the choice of system (1.13), to introduce "action-angle" variables $q_i = \sqrt{2I_i}\cos\varphi_i$, $p_i = \sqrt{2I_i}\sin\varphi_i$, $i = 1, 2$. The Hamiltonian in (1.13) takes the form

$$H = I_1 - I_2 - 4(I_1 + I_2)\sqrt{I_1 I_2}\cos(\varphi_1 + \varphi_2)\,,$$

which cannot be transformed into Birkhoff normal form because the frequency vector $\omega = (1, -1)$ of the harmonic oscillators "enters in resonance" with $\cos(k \cdot \varphi)$, $k = (1, 1)$ (resonance of order 2); see [10].

We also remark that the quadratic part of the Hamiltonian (1.13),

$$H_2 := \frac{q_1^2 + p_1^2}{2} - \frac{q_2^2 + p_2^2}{2}\,,$$

is neither positive nor negative definite, and that its signature satisfies

$$\mathrm{sign}(H_2) = 0$$

(the signature is the difference between the dimension of the maximal subspace where H_2 is positive definite and the dimension of the maximal subspace where H_2 is negative definite).

In contrast, two remarkable theorems by Weinstein [129]–Moser [98] and Fadell–Rabinowitz [61] prove the existence of periodic solutions under the respective assumptions

$$\text{(WM)}\ \ (D^2H)(0) > 0\ \ (\text{or}\ < 0\,)\,, \qquad \text{(FR)}\ \ \mathrm{sign}(D^2H)(0) \neq 0\,.$$

Note that (WM) is a stronger condition than (FR) and that the nonexistence example (1.13) violates the condition (FR).

Under (WM), the Weinstein–Moser theorem finds bifurcation of periodic solutions of the Hamiltonian system (HS) with fixed *energy*, while assuming (FR), the Fadell–Rabinowitz theorem proves the existence of solutions with fixed *period*.

Remark 1.11. Assuming just (FR), bifurcation of periodic solutions with fixed energy was more recently proved by Bartsch [16].

Let us give the precise statements of these theorems.

Theorem 1.12. (Weinstein 1973–Moser 1976) *Let $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$ be such that $(\nabla H)(0) = 0$ and $(D^2H)(0) > 0$. For all ε small enough there exist on each energy surface $\{H(x) = H(0) + \varepsilon^2\}$ at least n geometrically distinct periodic solutions of the Hamiltonian system (HS); see Figure 1.5.*

Remark 1.13. Theorem 1.12 was first proved by Weinstein [129]. Moser [98] proved the extensions described in Remarks 1.19 and 1.16 below.

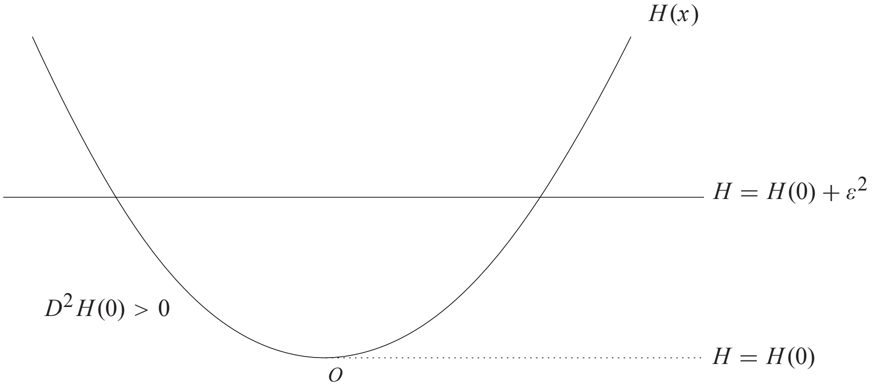

Fig. 1.5. The Weinstein–Moser theorem.

By the assumptions of the Weinstein–Moser theorem, Theorem 1.12, the point $x = 0$ is an elliptic equilibrium. Indeed, the Hamiltonian of the linearized system

$$\dot{x} = J(D^2H)(0)x$$

is

$$H_2 := \frac{1}{2}(D^2H)(0)x \cdot x\,.$$

Since the level sets of H_2 are ellipsoids, all the solutions of the linearized system are bounded, and therefore, the eigenvalues of the matrix $J(D^2H)(0)$ are all purely imaginary, say $\pm \mathrm{i}\omega_1, \ldots, \pm \mathrm{i}\omega_n$.

Remark 1.14. Since $(D^2H)(0) > 0$, it could be also proved (see, e.g., Theorem 8 in [72]) that via a linear symplectic change of variables, H_2 assumes the diagonal form

$$H_2 = \sum_{j=1}^{n} \omega_j \frac{q_j^2 + p_j^2}{2}\,. \tag{1.14}$$

The number of solutions found in Theorem 1.12 is, in general, optimal.

Remark 1.15. (**Optimality**) A first trivial example is provided by the quadratic Hamiltonian (1.14) where the frequency vector $\omega := (\omega_1, \ldots, \omega_n)$ of the decoupled harmonic oscillators satisfy the nonresonance condition

$$\omega \cdot k \neq 0\,, \qquad \forall k \in \mathbf{Z}^n \setminus 0\,. \tag{1.15}$$

In this case the only periodic solutions of (HS) have the form

$$\Big\{(q_j, p_j) = A_j(\cos(\omega_j t + \varphi_j), \sin(\omega_j t + \varphi_j))\,, \quad q_i = p_i = 0\,, \forall i \neq j\Big\}$$

for $j = 1, \ldots, n$, and A_j is uniquely fixed by the energy level.

However, the previous example is not really interesting: under the nonresonance condition (1.15), bifurcation of periodic solutions can be proved just by the Lyapunov center theorem as discussed in Remark 1.8.

There is also the following "completely resonant" example (for $n = 2$):

$$H = \frac{(q_1^2 + p_1^2) + (q_2^2 + p_2^2)}{1 + (q_1^2 + p_1^2) - (q_2^2 + p_2^2)},$$

where $(D^2 H)(0) > 0$ and the eigenvalues of the linearized system are all equal to $\pm 2\mathrm{i}$. Consider an energy level $\{H = h\}$. It is described also as

$$(q_1^2 + p_1^2)(1 - h) + (q_2^2 + p_2^2)(1 + h) = h\,. \tag{1.16}$$

In other words, the energy surface $\{H = h\}$ is also the energy surface of the quadratic Hamiltonian in the left-hand side of (1.16). But if two Hamiltonians share the same energy level, they possess (on this level) the same trajectories up to a reparametrization of time.[2] Then all the orbits of the Hamiltonian system generated by H on the surface $\{H = h\}$ are harmonic oscillations, and if

$$\frac{1-h}{1+h} \notin \mathbf{Q}\,,$$

then the system has exactly only $n = 2$ periodic orbits.

Remark 1.16. Moser [98] proved also an analogue of Theorem 1.12 for non-Hamiltonian systems: if (1.1) has a first integral G with $(D^2 G)(0) > 0$, then there exists at least one periodic solution on each level $\{G(x) = G(0) + \varepsilon^2\}$. The difference with respect to the Lyapunov center theorem, Theorem 1.6, is that the nonresonance condition (1.9) is NOT assumed.

We now describe the Fadell–Rabinowitz resonant center theorem.

Theorem 1.17. (Fadell–Rabinowitz 1978) *Let* $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$ *with* $(\nabla H)(0) = 0$. *Assume that any nonzero solution of the linearized system* $\dot{x} = J(D^2 H)(0)x$ *is* T*-periodic (and not constant). If*

$$\operatorname{sign}(D^2 H)(0) = 2\nu \neq 0\,,$$

then either

(*i*) $x = 0$ *is a nonisolated* T*-periodic solution of the Hamiltonian system (HS),*

or

[2] Assume $M = \{H(x) = h\} = \{F(x) = f\}$ with $\nabla H(x) \neq 0$, $\nabla F(x) \neq 0$ on M. Then $\nabla F(x) = \lambda(x)\nabla H(x)$ with $\lambda(x) \neq 0$ on M. If ϕ^t denotes the flow of the Hamiltonian vector field $J\nabla H(x)$, and ψ^s the flow of $J\nabla F(x)$, then we have, on M, $\psi^s(x) = \phi^t(x)$, where $t(s, x)$ solves $\mathrm{d}t/\mathrm{d}s = \lambda(\phi^t(x))$, $t(0, x) = 0$.

(*ii*) *there exist a pair of integers* $k, m \geq 0$ *with* $k+m \geq |\nu|$ *and a left neighborhood,* $\mathcal{U}_l$*, resp. a right neighborhood,* $\mathcal{U}_r$*, of* T *in* $\mathbf{R}$ *such that* $\forall \lambda \in \mathcal{U}_l \setminus \{T\}$*, resp.* $\mathcal{U}_r \setminus \{T\}$*, there exist at least k, resp. m, distinct, nontrivial,* λ*-periodic solutions of the Hamiltonian system (HS). The* L^∞*-norm of the solutions tends to* 0 *as* $\lambda \to T$*.*

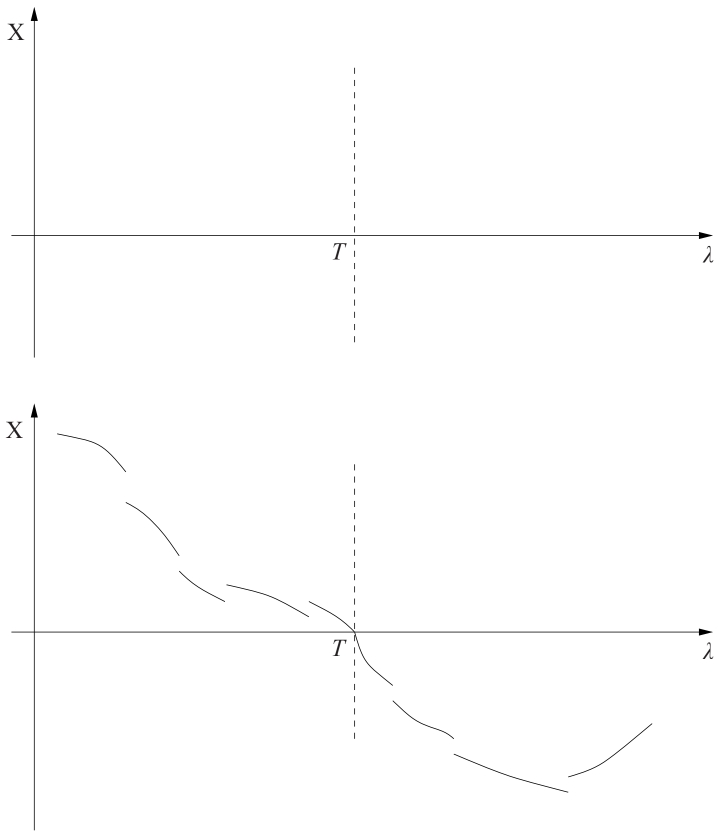

Fig. 1.6. The bifurcation diagram of the Fadell–Rabinowitz theorem. The upper diagram represents case (*i*): there exists a sequence of nontrivial periodic solutions of (HS) tending to 0 with period equal to T (this case holds, for example, if $H \equiv H(0) + (1/2)(D^2H)(0)x \cdot x$ is quadratic). In case (*ii*) there exist nontrivial periodic solutions with period λ near T. Such solutions of (HS) could not depend continuously with respect to the period λ.

Unlike the Lyapunov center theorem (see Remark 1.7), the periodic solutions of Theorems 1.12 and 1.17 do not, in general, vary smoothly with respect to the parameters ε (energy) and λ (period).

This will cause a serious difficulty for infinite-dimensional PDEs; see Chapter 4 and especially Remark 4.28.

Remark 1.18. For the extension to Hamiltonian PDEs, in light of the small-divisor problem discussed in the next chapters, it will be more convenient to count the number of periodic solutions with a given frequency, as in the Fadell–Rabinowitz theorem, and not with a given energy.

Remark 1.19. In the Weinstein–Moser theorem, resp. Fadell–Rabinowitz, one could assume a weaker condition. Suppose $\mathbf{R}^{2n} = E \oplus F$, where E and F are invariant subspaces of $\dot{x} = J(D^2 H)(0)x$, and all its solutions in E are T-periodic, while none of its solutions in $F \setminus \{0\}$ have this period. To conclude the existence of periodic solutions it is sufficient that $(D^2 H)(0)_{|E} > 0$ (Weinstein–Moser), respectively $\mathrm{sign}(D^2 H)(0)_{|E} \neq 0$ (Fadell–Rabinowitz).

We shall prove a simplified version of the theorems of Weinstein–Moser and Fadell–Rabinowitz without obtaining the optimal multiplicity results:

Theorem 1.20. (Weinstein–Moser) *Let $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$, $(\nabla H)(0) = 0$ and $(D^2 H)(0) > 0$. For all ε small enough there exist, on each energy surface $\{H(x) = H(0) + \varepsilon^2\}$, at least two geometrically distinct periodic solutions of the Hamiltonian system (HS).*

We shall prove also a simpler version of the Fadell–Rabinowitz theorem assuming that $(D^2 H)(0) > 0$. In this case the eigenvalues of the linearized Hamiltonian vector field $J(D^2 H)(0)$,

$$\pm \mathrm{i}\omega_1, \ldots, \pm \mathrm{i}\omega_n ,$$

are purely imaginary. For definiteness we suppose

$$\omega_1 = \omega_2 = \cdots = \omega_n = 1 , \tag{1.17}$$

so that the linearized system

$$\dot{x} = J(D^2 H)(0)x \tag{1.18}$$

possesses a $2n$-dimensional linear space of periodic solutions with the same minimal period 2π.

We remark, however, that, in the general Fadell–Rabinowitz theorem, Theorem 1.17, the periodic solutions of the linearized equation (1.18) need NOT have the same minimal period T. For example, it could be $\omega_j = j \in \mathbf{N}$, $j = 1, \ldots, n$, as for the completely resonant nonlinear wave equations studied in the next Chapter; see (2.6).

Theorem 1.21. (Fadell–Rabinowitz) *Under the assumptions above, either*

(i) *$x = 0$ is a nonisolated 2π-periodic solution of (HS);*

or

(ii) *there is a one-sided neighborhood $\mathcal{U}$ of 1 such that $\forall \lambda \in \mathcal{U} \setminus \{1\}$, the Hamiltonian system (HS) possesses at least two distinct nontrivial $2\pi\lambda$-periodic solutions;*

or

(iii) *there is a neighborhood* $\mathcal{U}$ *of* 1 *such that* $\forall \lambda \in \mathcal{U} \setminus \{1\}$*, the Hamiltonian system (HS) possesses at least one nontrivial* $2\pi\lambda$*-periodic solution.*

Theorem 1.21 is a particular case of a general bifurcation result for potential operators due to Rabinowitz [116]; see also [119].

Both the Weinstein–Moser theorem and the Fadell–Rabinowitz theorem follow by arguments of *variational bifurcation theory*.

The Weinstein–Moser theorem, thanks to the energy constraint, ultimately reduces to the search for critical points of a smooth functional defined on a $(2n-1)$-dimensional sphere, invariant under the S^1-action induced by time translations. Existence of a maximum and a minimum is obvious. Multiplicity of critical points is a consequence of the S^1-symmetry invariance; see, e.g., [119].

The Fadell–Rabinowitz theorem is more subtle because there is no energy constraint, and one has to look for critical points of a reduced-action functional defined in an *open* neighborhood of the origin.

1.2.1 The Variational Lyapunov–Schmidt Reduction

Suppose, for simplicity,

$$H(0) = 0 \quad \text{and} \quad (D^2H)(0) = I\,, \tag{1.19}$$

i.e., (1.17) (we shall prove also the Weinstein–Moser theorem in this resonant case, which contains the essence of the problem).

Normalizing the period, we look for 2π-periodic solutions of

$$J\dot{x} + \lambda \nabla H(x) = 0\,, \tag{1.20}$$

and hence $x(t/\lambda)$ is a $2\pi\lambda$-periodic solution of (HS).

Equation (1.20) is the Euler–Lagrange equation of the C^1-action functional

$$\Psi(\lambda, \cdot)\colon H^1(\mathbf{T}) \to \mathbf{R}\,, \qquad \mathbf{T} := (\mathbf{R}/2\pi\mathbf{Z})\,,$$

defined, on the standard Sobolev space of 2π-periodic functions $H^1(\mathbf{T})$, by

$$\Psi(\lambda, x) := \int_{\mathbf{T}} \Big(\frac{1}{2} J\dot{x}(t) \cdot x(t) + \lambda H(x(t))\Big) \mathrm{d}t\,. \tag{1.21}$$

Indeed, $\forall h \in H^1(\mathbf{T})$,

$$\begin{aligned}(D_x\Psi)(\lambda, x)\,[h] &= \int_{\mathbf{T}} \Big(\frac{1}{2} J\dot{x} \cdot h + \frac{1}{2} J\dot{h} \cdot x + \lambda \nabla H(x) \cdot h\Big)\, \mathrm{d}t \\ &= \int_{\mathbf{T}} \Big(J\dot{x} + \lambda \nabla H(x)\Big) \cdot h\, \mathrm{d}t\,, \end{aligned} \tag{1.22}$$

which follows after integration by parts and using the antisymmetry of J.

To find critical points of the highly indefinite functional Ψ we perform a Lyapunov–Schmidt reduction decomposing

$$H^1(\mathbf{T}) := V \oplus V^\perp ,$$

where

$$V := \left\{ v \in H^1(\mathbf{T}) \mid \dot v = J(D^2 H)(0) v \right\}$$

is a $2n$-dimensional linear space (by (1.17)) and

$$V^\perp := \left\{ w \in H^1(\mathbf{T}) \mid \int_{\mathbf{T}} w \cdot v \, \mathrm{d}t = 0 , \ \forall v \in V \right\} .$$

The map $V \ni v \to v(0) \in \mathbf{R}^{2n}$ is an isomorphism, and on the finite-dimensional space V, the H^1-norm $\|v\|_{H^1}$ is equivalent to the Euclidean norm $|v(0)|$, and it will be denoted by $\|v\|$.

Projecting (1.20) for $x = v + w$, $v \in V$, $w \in V^\perp$, yields

$$\begin{cases} \Pi_V (J(\dot v + \dot w) + \lambda \nabla H(v+w)) = 0 & \textit{bifurcation equation}, \\ \Pi_{V^\perp} (J(\dot v + \dot w) + \lambda \nabla H(v+w)) = 0 & \textit{range equation}, \end{cases}$$

where Π_V, $\Pi_{V^\perp}$ denote the projectors on V, resp. $V^\perp$.

1.2.2 Solution of the Range Equation

We solve first the range equation with the standard implicit function theorem, obtaining a solution $w(\lambda, v) \in V^\perp$ for $v \in B_r(0)$ ($\equiv$ ball in V of radius r centered at zero), $r > 0$ small enough, and λ sufficiently close to 1.

Indeed, the nonlinear operator

$$\mathcal{F} : \mathbf{R} \times V \times V^\perp \to \bar V^\perp$$

with range in $\bar V^\perp := \{ w \in L^2(\mathbf{T}) \mid \int_{\mathbf{T}} w \cdot v \, \mathrm{d}t = 0 , \ \forall v \in V \}$, defined by

$$\mathcal{F}(\lambda, v, w) := \Pi_{V^\perp} (J(\dot v + \dot w) + \lambda \nabla H(v+w)) ,$$

vanishes at $\mathcal{F}(1,0,0) = 0$, and its partial derivative

$$(D_w \mathcal{F})(1,0,0)[W] = \Pi_{V^\perp}(J \dot W + (D^2 H)(0) W) , \quad \forall W \in V^\perp ,$$

is an isomorphism between $V^\perp$ and $\bar V^\perp$. The solution $w(\lambda, v) \in V^\perp$ of the range equation is a C^1 function with respect to (λ, v).

We have

$$w(\lambda, 0) = 0 \tag{1.23}$$

and

$$(D_v w)(1,0) = -(D_w \mathcal{F})^{-1}(1,0,0)(D_v \mathcal{F})(1,0,0) = 0 \tag{1.24}$$

because

$$(D_v \mathcal{F})(1,0,0)\,[h] = \Pi_{V^\perp}(J\dot{h} + (D^2 H)(0)h) = 0\,, \quad \forall h \in V\,.$$

By (1.23)–(1.24) it follows that

$$w(\lambda, v) = o(\|v\|) \qquad \text{as } v \to 0 \tag{1.25}$$

uniformly for λ near 1.

For future reference we also note that

$$\|\partial_\lambda w(\lambda, v)\| = O(\|v\|) \qquad \text{as } v \to 0 \tag{1.26}$$

for λ close to 1. Indeed, differentiating yields

$$\begin{aligned}(D_w \mathcal{F})(\lambda, v, w(\lambda, v))\,\partial_\lambda w(\lambda, v) &= -(\partial_\lambda \mathcal{F})(\lambda, v, w(\lambda, v))\\ &= -\Pi_{V^\perp} \nabla H(v + w(\lambda, v))\,.\end{aligned} \tag{1.27}$$

For $\lambda \to 1$, $v \to 0$, $(D_w \mathcal{F})(\lambda, v, w(\lambda, v))$ is an isomorphism, being close to $(D_w \mathcal{F})(1,0,0)$. By (1.27), (1.25) and since $\nabla H(0) = 0$, we deduce (1.26).

Remark 1.22. Under the assumption (1.19), $\partial_\lambda w(\lambda, v) = o(\|v\|)$, because

$$\Pi_{V^\perp} \nabla H(v + w(\lambda, v)) = \Pi_{V^\perp}(v + w(\lambda, v)) + o(\|v\|) = w(\lambda, v) + o(\|v\|).$$

1.2.3 Solution of the Bifurcation Equation

It remains to solve the *finite-dimensional* bifurcation equation

$$\Pi_V(J(\dot{v} + \dot{w}(\lambda, v)) + \lambda \nabla H(v + w(\lambda, v))) = 0\,. \tag{1.28}$$

Equation (1.28) is still variational, being the Euler–Lagrange equation of the "reduced-action functional" $\Phi(\lambda, \cdot) \colon B_r(0) \subset V \to \mathbf{R}$ defined by

$$\Phi(\lambda, v) := \Psi(\lambda, v + w(\lambda, v))\,. \tag{1.29}$$

Indeed, $\forall h \in V$,

$$\begin{aligned}(D_v \Phi)(\lambda, v)\,[h] &= (D_x \Psi)(\lambda, v + w(\lambda, v))\Big[h + (D_v w)(\lambda, v)\,[h]\Big]\\ &= (D_x \Psi)(\lambda, v + w(\lambda, v))\,[h] && (1.30)\\ &\overset{(1.22)}{=} \int_{\mathbf{T}} \Pi_V(J(\dot{v} + \dot{w}) + \lambda \nabla H(v + w)) \cdot h\,, && (1.31)\end{aligned}$$

and in (1.30) we have used that

$$(D_x \Psi)(\lambda, v + w(\lambda, v))\,[W] = 0\,, \qquad \forall W \in V^\perp\,, \tag{1.32}$$

since $w(\lambda, v)$ solves the range equation, and $(D_v w)(\lambda, v)\,[h] \in V^\perp$.

Remark 1.23. The variational nature of the bifurcation equation dealing with variational problems is a general fact; see, e.g., [4], [119].

Now the proofs of the Weinstein–Moser and Fadell–Rabinowitz resonant center theorems become different.

1.2.4 Proof of the Weinstein–Moser Theorem

We need to find critical points of the action functional

$$\Psi(\lambda, x) := \int_{\mathbf{T}} \frac{1}{2} J\dot{x}(t) \cdot x(t) + \lambda \int_{\mathbf{T}} (H(x(t)) - \varepsilon^2)\, \mathrm{d}t\,, \tag{1.33}$$

which differs just for the "energy" constant ε^2 by the action functional (1.21); with a slight abuse of notation we shall use the same symbol for denoting both (clearly, their corresponding Euler–Lagrange equations are the same).

We do *not* fix the value of λ, but we impose that

$$(D_v \Phi)(\lambda, v)\,[v] = 0 \tag{1.34}$$

(the radial derivative of the reduced-action functional Φ vanishes).

Lemma 1.24. *Equation (1.34) can be solved for* $\lambda = \lambda(v)$, $\forall v \in B_r(0) \setminus \{0\}$, *for* $r > 0$ *small enough. The function* $\lambda(v)$ *is* C^1, $\lambda(v) \to 1$ *as* $v \to 0$, *and*

$$D_v \lambda(v)\,[v] \to 0 \quad \text{as} \quad v \to 0\,. \tag{1.35}$$

Postponing the proof of Lemma 1.24, we now consider

$$\mathcal{S}_\varepsilon := \left\{ v \in B_r(0) \;\middle|\; S(v) := \int_{\mathbf{T}} H(v + w(\lambda(v), v))\, \mathrm{d}t = 2\pi\varepsilon^2 \right\}$$

for $\varepsilon > 0$ small enough. Since $S(0) = 0$, then $v = 0 \notin \mathcal{S}_\varepsilon$.

Lemma 1.25. $\mathcal{S}_\varepsilon \subset B_r(0) \setminus \{0\}$ *is a compact (sphere-like) manifold and*

$$T_v \mathcal{S}_\varepsilon \oplus \langle v \rangle = V\,, \qquad \forall v \in \mathcal{S}_\varepsilon\,. \tag{1.36}$$

Proof. We claim that there exist $c_1, c_2 > 0$ such that

$$c_1 \|v\|^2 \le S(v) \le c_2 \|v\|^2\,, \quad c_1 \|v\|^2 \le D_v S(v)\,[v] \le c_2 \|v\|^2 \tag{1.37}$$

as $v \to 0$. Indeed, since $H(0) = 0$, $\nabla H(0) = 0$, and by (1.25),

$$S(v) := \int_{\mathbf{T}} H(v + w(\lambda(v), v)) = \frac{1}{2} \int_{\mathbf{T}} (D^2 H)(0) v \cdot v + o(\|v\|^2)\,,$$

whence the first inequalities in (1.37) (we use that $(D^2 H)(0) v \cdot v$ is constant along the solutions v of the linearized system (1.18)).

Next, using (1.25), (1.26), (1.24), and (1.35),

$$\begin{aligned} D_v S(v)\,[v] &= \int_{\mathbf{T}} (\nabla H)(v + w(\lambda(v), v)) \Big[v + (D_v w)\,[v] + (D_\lambda w) D_v \lambda(v)\,[v] \Big] \\ &= \int_{\mathbf{T}} (D^2 H)(0)\,[v + o(\|v\|)]\,[v + o(\|v\|)]\,, \end{aligned} \tag{1.38}$$

whence the second inequalities in (1.37).

By the first inequalities in (1.37), $\mathcal{S}_\varepsilon$ is strictly contained between the spheres of radii $\varepsilon(2\pi/c_2)^{1/2}$ and $\varepsilon(2\pi/c_1)^{1/2}$. Moreover, $\mathcal{S}_\varepsilon$ is closed because S is continuous. Furthermore, by the second inequalities in (1.37), $\forall v \in \mathcal{S}_\varepsilon$, $D_v S(v) \neq 0$, whence $\mathcal{S}_\varepsilon$ is a manifold with tangent space

$$T_v \mathcal{S}_\varepsilon = \Big\{ h \in V \mid D_v S(v)\,[h] = 0 \Big\} ,$$

and (1.36) follows because $D_v S(v)\,[v] > 0$ by (1.37). ■

Remark 1.26. Under (1.19), $D_\lambda w = o(\|v\|)$ by Remark 1.22, and we would not need the estimate (1.35) to get (1.38). We have preferred, however, to give the general argument.

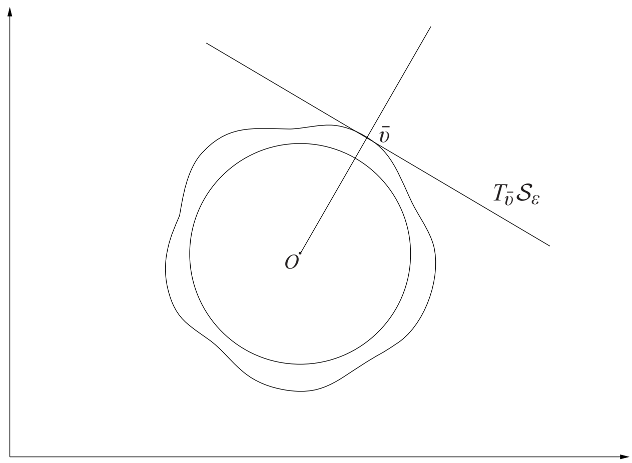

Fig. 1.7. The sphere-like manifold $\mathcal{S}_\varepsilon$.

Finally, we define the functional $I\colon B_r(0) \to \mathbf{R}$ as

$$I(v) := \Phi(\lambda(v), v)\,.$$

Lemma 1.27. *A critical point $\bar v \in \mathcal{S}_\varepsilon$ of $I\colon \mathcal{S}_\varepsilon \to \mathbf{R}$ is a critical point of*

$$\Phi(\lambda(\bar v), \cdot)\colon B_r(0) \subset V \to \mathbf{R}$$

(with fixed period $\lambda(\bar v)$). As a consequence,

$$\bar y(t) = \bar x\left(\frac{t}{\lambda(\bar v)}\right), \qquad \text{where} \qquad \bar x := \bar v + w(\lambda(\bar v), \bar v)$$

is a $2\pi\,\lambda(\bar v)$-periodic solution of (HS) with energy $H(\bar y) = \varepsilon^2$.

Proof. Differentiating in the definition of I yields, $\forall h \in T_{\bar{v}}\mathcal{S}_\varepsilon$,

$$DI(\bar{v})[h] = (\partial_\lambda \Phi)(\lambda(\bar{v}), \bar{v}) D_v \lambda(\bar{v})[h] + (\partial_v \Phi)(\lambda(\bar{v}), \bar{v})[h] \,. \tag{1.39}$$

Now

$$\begin{aligned}(\partial_\lambda \Phi)(\lambda, \bar{v}) &\overset{(1.29)}{=} (\partial_\lambda \Psi)(\lambda, \bar{v} + w(\lambda, \bar{v})) + (D_x \Psi)(\lambda, \bar{v} + w(\lambda, \bar{v}))[\partial_\lambda w] \\ &\overset{(1.33)}{=} \int_{\mathbf{T}} (H(\bar{v} + w(\lambda, \bar{v})) - \varepsilon^2)\, \mathrm{d}t \end{aligned} \tag{1.40}$$

because $(D_x \Psi)(\lambda, \bar{v} + w(\lambda, \bar{v}))[\partial_\lambda w] = 0$ by (1.32) and $\partial_\lambda w \in V^\perp$.

Furthermore,

$$(\partial_\lambda \Phi)(\lambda(\bar{v}), \bar{v}) = 0$$

because the term (1.40) vanishes for $\lambda := \lambda(\bar{v})$, since $\bar{v} \in \mathcal{S}_\varepsilon$. We conclude by (1.39) that at the constrained critical point $\bar{v} \in \mathcal{S}_\varepsilon$ of I,

$$0 = DI(\bar{v})[h] = (\partial_v \Phi)(\lambda(\bar{v}), \bar{v})[h] \,, \qquad \forall h \in T_{\bar{v}}\mathcal{S}_\varepsilon \,. \tag{1.41}$$

By (1.34) we have also $(\partial_v \Phi)(\lambda(\bar{v}), \bar{v})[\bar{v}] = 0$, and since $T_{\bar{v}}\mathcal{S}_\varepsilon \oplus \langle \bar{v} \rangle = V$, we deduce that $\bar{v}$ is a critical point of $\Phi(\lambda(\bar{v}), \cdot) \colon B_r(0) \subset V \to \mathbf{R}$.

Hence $\bar{x} = \bar{v} + w(\lambda(\bar{v}), \bar{v})$ is a solution of (1.20) and its energy $H(\bar{v} + w(\lambda(\bar{v}), \bar{v}))(t)$ is constant in time. Therefore, since $\bar{v} \in \mathcal{S}_\varepsilon$, it is equal to ε^2. ■

To conclude the proof of Theorem 1.20, since the manifold $\mathcal{S}_\varepsilon$ is *compact*, $I_{|S_\varepsilon}$ possesses at least one maximum and one minimum, implying the existence of at least two geometrically distinct periodic solutions of the Hamiltonian system (HS) with energy ε^2 and with periods close to 2π.

Proof of Lemma 1.24. By (1.31),

$$\begin{aligned}(D_v \Phi)(\lambda, v)[v] &\overset{(1.31)}{=} \int_{\mathbf{T}} J(\dot{v} + \dot{w}(\lambda, v)) \cdot v + \lambda \nabla H(v + w(\lambda, v)) \cdot v \\ &= \mathcal{R}(\lambda, v) + (\lambda - 1) Q(\lambda, v) \,, \end{aligned} \tag{1.42}$$

where

$$Q(\lambda, v) := \int_{\mathbf{T}} (\nabla H)(v + w(\lambda, v)) \cdot v \,,$$

and since $J\dot{v} = -(D^2 H)(0)v$, using the antisymmetry of J and integrating by parts yields

$$\begin{aligned}\mathcal{R}(\lambda, v) &:= \int_{\mathbf{T}} J\dot{w} \cdot v + \nabla H(v + w) \cdot v - (D^2 H)(0)v \cdot v \\ &= \int_{\mathbf{T}} w \cdot J\dot{v} + \nabla H(v + w) \cdot v - (D^2 H)(0)v \cdot v \\ &= \int_{\mathbf{T}} \Big[\nabla H(v + w) - (D^2 H)(0)(v + w) \Big] \cdot v \,. \end{aligned} \tag{1.43}$$

By (1.42), equation (1.34) amounts to solving

$$\lambda = 1 - \frac{\mathcal{R}(\lambda, v)}{Q(\lambda, v)} . \tag{1.44}$$

By (1.43), since $(\nabla H)(0) = 0$ and by (1.25),

$$\mathcal{R}(\lambda, v) = o(\|v\|^2) , \tag{1.45}$$

and differentiating and using also (1.26),

$$\partial_\lambda \mathcal{R}(\lambda, v) = \int_{\mathbf{T}} \Big[(D^2 H)(v + w) - (D^2 H)(0) \Big] \partial_\lambda w \cdot v = o(\|v\|^2) . \tag{1.46}$$

Next, still by (1.25),

$$Q(\lambda, v) = \int_{\mathbf{T}} (D^2 H)(0) v \cdot v + o(\|v\|^2) \geq c \|v\|^2 \tag{1.47}$$

for some constant $c > 0$ and v small enough.

By (1.45) and (1.47), we have

$$\frac{\mathcal{R}(\lambda, v)}{Q(\lambda, v)} \to 0 \qquad \text{as} \qquad v \to 0 .$$

Furthermore,

$$\partial_\lambda \Big(\frac{\mathcal{R}(\lambda, v)}{Q(\lambda, v)} \Big) = \frac{\partial_\lambda \mathcal{R}(\lambda, v)}{Q(\lambda, v)} - \frac{\mathcal{R}(\lambda, v) \, \partial_\lambda Q(\lambda, v)}{(Q(\lambda, v))^2} . \tag{1.48}$$

Differentiating in the definition of Q yields

$$\partial_\lambda Q(\lambda, v) := \int_{\mathbf{T}} (D^2 H)(v + w(\lambda, v)) \partial_\lambda w \ \cdot v \overset{(1.26)}{=} O(\|v\|^2) . \tag{1.49}$$

By (1.48), the estimates (1.46), (1.47), (1.45), and (1.49) imply

$$\partial_\lambda \Big(\frac{\mathcal{R}(\lambda, v)}{Q(\lambda, v)} \Big) \to 0 \qquad \text{as} \qquad v \to 0 . \tag{1.50}$$

By an implicit function theorem argument we can solve (1.44), obtaining $\lambda(v)$ for v small. Clearly $\lambda(v) \to 1$ as $v \to 0$.

We finally prove (1.35). Differentiating in (1.44) yields

$$D_v \lambda(v) \, [v] = - \frac{(D_v G)(\lambda(v), v) \, [v]}{1 + (D_\lambda G)(\lambda(v), v)} ,$$

where for brevity we have set $G(\lambda, v) := \mathcal{R}(\lambda, v) / Q(\lambda, v)$.

By (1.50), $D_\lambda G(\lambda, v) \to 0$ as $v \to 0$, whence

$$|D_v \lambda(v) \, [v] \, | \leq 2 |(D_v G)(\lambda(v), v) \, [v] \, | . \tag{1.51}$$

Finally (omitting the symbol of $\lambda(v)$),

$$(D_v G)(v)[v] = \frac{D_v \mathcal{R}(v)[v]}{Q(v)} - \frac{\mathcal{R}(v)\, D_v Q(v)[v]}{(Q(v))^2} .$$

Now, differentiating in (1.43),

$$\begin{aligned} D_v \mathcal{R}(v)[v] = \int_{\mathbf{T}} &\Big[(D^2 H)(v+w) - (D^2 H)(0)\Big](v + D_v w[v]) \cdot v \\ &+ \Big[(\nabla H)(v+w) - (D^2 H)(0)(v+w)\Big] \cdot v = o(\|v\|^2) \end{aligned}$$

and

$$(D_v Q)(v)[v] = \int_{\mathbf{T}} (D^2 H)(v+w)\Big[v + (D_v w)(\lambda(v), v)[v]\Big] \cdot v = O(\|v\|^2)$$

imply, together with (1.47) and (1.45), that $(D_v G)(v)[v] \to 0$ as $v \to 0$. By (1.51) we deduce (1.35). ■

1.2.5 Proof of the Fadell–Rabinowitz Theorem

To prove Theorem 1.21 we have to find nontrivial critical points of the reduced action functional $\Phi(\lambda, \cdot)$ defined in (1.29) near $v = 0$ for fixed λ near 1 (fixed period).

The functional $\Phi(\lambda, \cdot)$ possesses a strict local minimum or maximum at $v = 0$, according as $\lambda > 1$ or $\lambda < 1$, because by (1.25) and $H(0) = \nabla H(0) = 0$,

$$\begin{aligned} \Phi(\lambda, v) &= \int_{\mathbf{T}} \frac{1}{2} J\dot{v} \cdot v + \frac{\lambda}{2}(D^2 H)(0)v \cdot v + o(\|v\|^2) \\ &= \frac{(\lambda - 1)}{2} \int_0^{2\pi} (D^2 H)(0)v(t) \cdot v(t)\, \mathrm{d}t + o(\|v\|^2) \\ &= \frac{(\lambda - 1)}{2} 2\pi (D^2 H)(0)v(\theta) \cdot v(\theta) + o(\|v\|^2), \quad \forall \theta \in \mathbf{T}, \quad (1.52) \end{aligned}$$

because $(D^2 H)(0)v \cdot v$ is constant along the solutions of (1.18).

If $v = 0$ is *not* an isolated critical point of $\Phi(1, \cdot)$, then alternative (i) of Theorem 1.21 holds.

Thus, for what follows, we assume that $v = 0$ is an isolated critical point of $\Phi(1, \cdot)$. Consequently, either

(a) $v = 0$ is a strict local maximum or minimum for $\Phi(1, \cdot)$;
(b) $\Phi(1, \cdot)$ takes on both positive and negative values near $v = 0$.

Case (a) leads to alternative (ii) and Case (b) leads to alternative (iii) of Theorem 1.21. Figure 1.8 gives the idea of the existence proof.

Case (a): Suppose $v = 0$ is a strict local maximum of $\Phi(1, \cdot)$ (to handle the case of a strict local minimum just replace Φ with $-\Phi$).

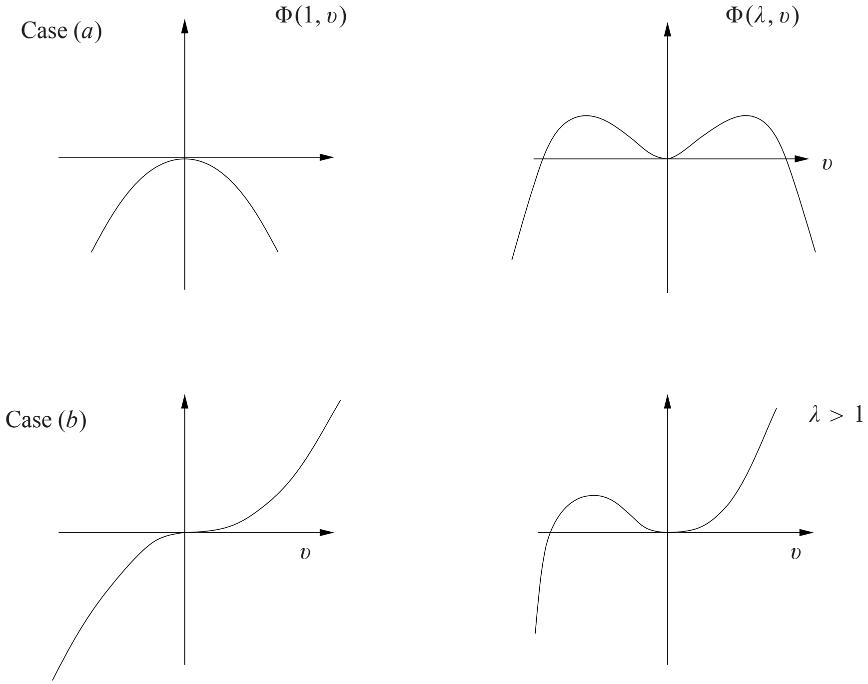

Fig. 1.8. In case (a), $\Phi(1,\cdot)$ has a strict local maximum at $\upsilon = 0$, and for $\lambda > 1$, $\Phi(\lambda,\cdot)$ possesses at least two nontrivial critical points. In case (b), $\Phi(1,\cdot)$ takes on both positive and negative values near $\upsilon = 0$, and for $\lambda \neq 1$, $\Phi(\lambda,\cdot)$ possesses at least one nontrivial critical point.

Since $\Phi(1,0) = 0$, for $r > 0$ small enough,

$$\exists \beta > 0 \quad \text{such that} \quad \Phi_{|\partial B_r}(1,\cdot) \leq -2\beta\,.$$

By continuity, for λ near 1,

$$\Phi_{|\partial B_r}(\lambda,\cdot) \leq -\beta\,.$$

Take $\lambda > 1$. By (1.52), $\Phi(\lambda,\cdot)$ has a local minimum at $\upsilon = 0$, and there exist $\rho \in (0,r)$, $\alpha(\lambda) > 0$ such that

$$\Phi(\lambda,\upsilon) \geq \alpha(\lambda) > 0\,, \qquad \forall \upsilon \in \partial B_\rho\,; \tag{1.53}$$

see Figure 1.9.

The maximum

$$\overline{c}(\lambda) := \max_{\upsilon \in \bar{B}_r} \Phi(\lambda,\upsilon) \geq \alpha(\lambda) > 0$$

is attained in the interior of B_r because $\Phi(\lambda,\cdot)$ is negative on ∂B_r. Furthermore, the maximum is not attained in $\upsilon = 0$ because $\Phi(\lambda,0) = 0$.

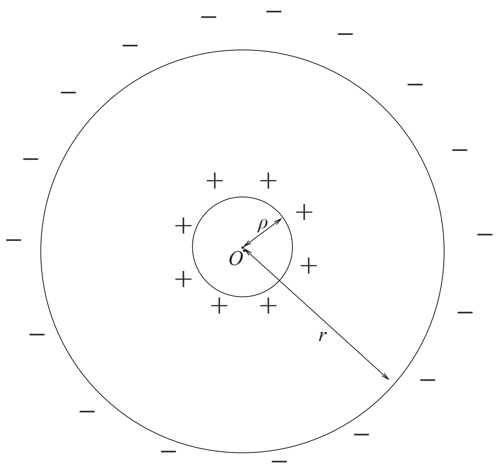

Fig. 1.9. Case (a): the level sets of the functional $\Phi(\lambda,\cdot)$.

To find another nontrivial critical point of $\Phi(\lambda,\cdot)$, we define the mountain pass critical level

$$\underline{c}(\lambda) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\lambda,\gamma(t)),$$

where the minimax class Γ is

$$\Gamma := \left\{\gamma \in C([0,1],\bar{B}_r) \mid \gamma(0)=0 \text{ and } \gamma(1)\in\partial B_r\right\}. \tag{1.54}$$

Since any path $\gamma \in \Gamma$ intersects $\{v\in V \mid \|v\|=\rho\}$, by (1.53),

$$\underline{c}(\lambda) \geq \alpha(\lambda) > 0. \tag{1.55}$$

The mountain pass theorem of Ambrosetti–Rabinowitz [7] cannot be directly applied because $\Phi(\lambda,\cdot)$ is defined only in a neighborhood of 0. However, since

$$\underline{c}(\lambda) \geq \alpha(\lambda) > 0 > -\beta \geq \Phi(\lambda,\cdot)_{|\partial B_r},$$

it is easy to adapt its proof to show that $\underline{c}(\lambda)$ is a critical value. Applying Theorem B.10 in the appendix to $\Phi(\lambda,\cdot)$, we conclude that there exists a Palais–Smale sequence $v_n \in B_r$ at the level $\underline{c}(\lambda)$, i.e., such that

$$\Phi(\lambda,v_n) \to \underline{c}(\lambda), \quad \nabla\Phi(\lambda,v_n) \to 0. \tag{1.56}$$

By compactness, up to a subsequence, $v_n \to \bar{v} \in \bar{B}_r$. Actually, $\bar{v} \in B_r(0)\setminus\{0\}$ because $\Phi(\lambda,\bar{v}) = \underline{c}(\lambda) > 0$ and $\Phi(\lambda,\cdot)_{|\partial B_r} < 0$, $\Phi(\lambda,0)=0$. The point $\bar{v}$ is a nontrivial critical point of $\Phi(\lambda,\cdot)$ at the level $\underline{c}(\lambda)$.

If $\underline{c}(\lambda) < \overline{c}(\lambda)$, then $\Phi(\lambda,\cdot)$ has two distinct critical points. If $\underline{c}(\lambda) = \overline{c}(\lambda)$, then $\overline{c}(\lambda)$ equals the maximum of $\Phi(\lambda,\cdot)$ over every curve in Γ. Therefore there are

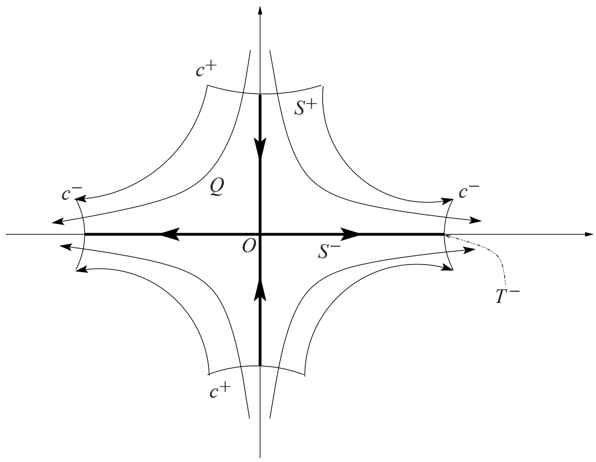

Fig. 1.10. The isolating "Conley set."

infinitely many maxima for $n \geq 2$, and at least 2 for $n = 1$. In any case, alternative (ii) holds.

Case (b). Again the idea is to reduce the proof to an argument of "mountain pass" type. In this case, since $\Phi(1, \cdot)$ takes on both positive and negative values near $v = 0$, the functional $\Phi(\lambda, \cdot)$ possesses the mountain-pass geometry both for $\lambda > 1$ and $\lambda < 1$. The construction is more subtle than the previous case because a level set of $\Phi(1, \cdot)$ no longer bounds a compact neighborhood of $v = 0$.

Let us consider the negative gradient flow

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\eta(t, v) = -\nabla\Phi(1, \eta(t, v)) \,, \\ \eta(0, v) = v \,, \end{cases} \tag{1.57}$$

which has a unique solution local in time for all $v \in V$ near zero (by (1.30) the vector field $\nabla\Phi(1, v)$ is in C^1 because $w(\lambda, v) \in C^1$).

In the sequel we take $r > 0$ small enough that $\Phi(1, \cdot)$ has no critical points in $\bar{B}_r$ except $v = 0$, i.e., the only rest point of $\nabla\Phi(1, \cdot)$ in $\bar{B}_r$ is $v = 0$.

In order to apply a deformation argument, the main issue is the construction of an "isolating Conley set":

Proposition 1.28. ([116]–[119]) *There exists an open neighborhood $Q \subset B_r$ of $v = 0$ and constants $c_+ > 0 > c_-$ such that $\forall v \in \partial Q$ either*

(i) $\Phi(1, v) = c^+$ *or*
(ii) $\Phi(1, v) = c^-$ *or*
(iii) $\eta(t, v) \in \partial Q$*,* $\forall t$ *near* 0.

Proof. The open neighborhood Q will be constructed taking a sufficiently small ball $B_\varepsilon \subset B_r$ and letting it evolve in the future and in the past by the negative gradient flow generated by (1.57); see precisely the definition (1.58).

We follow the exposition in [119]. Let us first define

$$S^+ := \left\{ v \in \bar{B}_r \;:\; \eta(t, v) \in \bar{B}_r \; \forall t > 0 \right\}$$

and

$$S^- := \left\{ v \in \bar{B}_r \;:\; \eta(t, v) \in \bar{B}_r \; \forall t < 0 \right\}.$$

Lemma 1.29. S^+, $S^- \neq \emptyset$.

Proof. Let us verify the S^+ case. Let $v_m \in B_r$ be a sequence such that $v_m \to 0$ and $\Phi(1, v_m) > 0$. Consider the solution $\eta(t, v_m)$ of (1.57).

We claim that there exists a largest $t_m < 0$ such that $\eta(t_m, v_m) =: y_m \in \partial B_r$. If not, the orbit defined for negative times passing through v_m at $t = 0$ is contained in B_r, so it must have a cluster point that is a critical point of $\Phi(1, \cdot)$ in $\bar{B}_r$ with positive critical value. But by assumption, there are no other critical points of $\Phi(1, \cdot)$ in $\bar{B}_r$ except $v = 0$.

Furthermore, $t_m \to -\infty$ as $m \to +\infty$, because $v = 0$ is an equilibrium of the C^1 vector field $-\nabla\Phi(1, v)$.

Let y be a limit point of y_m. Then $\eta(t, y) \in \bar{B}_r$ for all $t > 0$ and so $y \in S^+$. ∎

Remark 1.30. By the negative gradient structure of (1.57), there results

$$S^+ \equiv \left\{ v \in \bar{B}_r \;:\; \eta(t, v) \in \bar{B}_r \; \forall t > 0 \text{ and } \eta(t, v) \stackrel{t\to+\infty}{\longrightarrow} 0 \right\}.$$

Indeed, if $\eta(t, v)$ remains in $\bar{B}_r$ for all $t > 0$, $\eta(t, v)$ has to converge to some connected component of the critical set of $\Phi(1, \cdot)$ in $\bar{B}_r$, i.e., to $v = 0$, by the assumption on $r > 0$. Similarly,

$$S^- \equiv \left\{ v \in \bar{B}_r \;:\; \eta(t, v) \in \bar{B}_r \; \forall t < 0 \text{ and } \eta(t, v) \stackrel{t\to-\infty}{\longrightarrow} 0 \right\}.$$

Let us define $A_c := \{v \in V \;:\; \Phi(1, v) \le c\}$.

Lemma 1.31. *There exists* $0 < \varepsilon < r$ *such that if* $c := \max_{v \in \bar{B}_\varepsilon} \Phi(1, v)$ *and* $v \in \bar{B}_\varepsilon \backslash S^-$*, then as* $t \to -\infty$*, the orbit* $\eta(t, v)$ *leaves* $A_c \cap B_r$ *via* $A_c \backslash \partial B_r$.

Proof. By contradiction. There are sequences $\varepsilon_m \to 0$, $v_m \in \bar{B}_{\varepsilon_m} \backslash S^-$, and $t_m < 0$ such that $z_m := \eta(t_m, v_m) \in \partial B_r$ (we take t_m to be the largest time such that $\eta(t_m, v_m) \in \partial B_r$) and

$$\min_{v \in \bar{B}_{\varepsilon_m}} \Phi(1, v) \le \Phi(1, z_m) \le \max_{v \in \bar{B}_{\varepsilon_m}} \Phi(1, v).$$

Therefore, up to a subsequence, $z_m \to z \in \partial B_r$ and $\Phi(1, z) = 0$. Arguing as in the proof of Lemma 1.29, we deduce that $z \in S^+$. By Remark 1.30 we get $\eta(t, z) \to 0$ as $t \to +\infty$. Hence $\Phi(1, \eta(t, z)) \to \Phi(1, 0) \equiv 0$ as $t \to +\infty$. However, $\Phi(1, z) = 0$, and the function $t \to \Phi(1, \eta(t, z))$ is strictly decreasing. This contradiction proves the lemma. ∎

Let ε be as in Lemma 1.31 and define

$$c^+ := \max_{v \in \bar{B}_\varepsilon} \Phi(1, v) > 0\,, \qquad c^- := \min_{v \in \bar{B}_\varepsilon} \Phi(1, v) < 0\,.$$

By the previous lemma we get the following:

Lemma 1.32. $\forall v \in \bar{B}_\varepsilon \backslash S^-$ *there is a corresponding* $t^-(v) < 0$ *such that*

$$\Phi(1, \eta(t^-(v), v)) = c^+ \quad \textit{and} \quad \eta(t^-(v), v) \notin \partial B_r\,.$$

Similarly, $\forall v \in \bar{B}_\varepsilon \backslash S^+$ *there is a corresponding* $t^+(v) > 0$ *such that*

$$\Phi(1, \eta(t^+(v), v)) = c^- \quad \textit{and} \quad \eta(t^+(v), v) \notin \partial B_r\,.$$

Finally, define

$$Q := \left\{ \eta(t, v) \ : \ v \in B_\varepsilon \quad \text{and} \quad t^-(v) < t < t^+(v) \right\}\,, \tag{1.58}$$

where if $v \in S^-$, then $t^-(v) := -\infty$, and if $v \in S^+$, then $t^+(v) := +\infty$.

Lemma 1.33. *The function* $t^-(\cdot)\colon \bar{B}_\varepsilon \to \mathbf{R} \cup \{-\infty\}$ *is upper semicontinuous;* $t^+(\cdot)\colon \bar{B}_\varepsilon \to \mathbf{R} \cup \{+\infty\}$ *is lower semicontinuous.*

Proof. Let us verify the t^+ case.
(a) We claim that if $v \in \bar{B}_\varepsilon \backslash S^+$, then $\forall\, v_n \to v \Rightarrow \liminf t^+(v_n) \geq t^+(v)$.

If not, there exists $v_n \to v$, $v_n \in \bar{B}_\varepsilon \backslash S^+$, such that $\tau := \liminf t^+(v_n) < t^+(v)$. Since the function $t \to \Phi(1, \eta(t, v))$ is strictly decreasing, we get

$$\Phi(1, \eta(\tau, v)) > \Phi(1, \eta(t^+(v), v)) = c^-\,. \tag{1.59}$$

However, $c^- = \Phi(1, \eta(t^+(v_n), v_n))$, $\forall n$. Up to a subsequence, $t^+(v_n) \to \tau$, and therefore, by the continuity of the flow, we get

$$c^- = \lim \Phi(1, \eta(t^+(v_n), v_n)) = \Phi(1, \eta(\tau, v)) > c^-$$

by (1.59). Contradiction.

Actually, it is true that t^+ is continuous on $\bar{B}_\varepsilon \backslash S^+$.
(b) We claim that if $v \in \bar{B}_\varepsilon \cap S^+$, then $\forall\, v_n \to v \Rightarrow \lim t^+(v_n) = +\infty$.

If not, there exist $v_n \to v$ and $T < +\infty$ such that $t^+(v_n) < T$. Clearly $v_n \in \bar{B}_\varepsilon \backslash S^+$ (because $t^+(v) = +\infty$ on S^+).

We have $\Phi(1, \eta(t^+(v_n), v_n)) = c^- < 0$, $\forall n$. Up to a subsequence, $t^+(v_n) \to \tau \leq T$, and therefore, $\Phi(1, \eta(\tau, v)) = c^- < 0$. This contradicts the fact that $\Phi(1, \eta(t, v)) \geq 0$, $\forall t > 0$ (since $v \in S^+$, then $\eta(t, v) \to 0$ as $t \to +\infty$ and $t \to \Phi(1, \eta(t, v))$ is strictly decreasing). ■

Proof of Proposition 1.28 *completed.* We first verify that Q is an open neighborhood of $v = 0$. Clearly $0 \in B_\varepsilon \subset Q$. Next, if $z \in Q$ then $z = \eta(t, v)$ for some $v \in B_\varepsilon$ and some $t \in (t^-(v), t^+(v))$. Take δ small such that $B_\delta(v) \subset B_\varepsilon$ and such that for all $w \in B_\delta(v), t^-(w) < t < t^+(w)$ (which exists because $t^+(\cdot)$ is lower semicontinuous and $t^-(\cdot)$ is upper semicontinuous).

The set $\eta(t, B_\delta(v)) \subset Q$ is a neighborhood of $z \in Q$. Therefore Q is open.

Let us show that if $v \in \partial Q$ then either (i) or (ii) holds, or the possibility (iii) occurs.

Let $v \in \partial Q$. Hence there exist $v_n := \eta(t_n, x_n) \in Q$, $x_n \in B_\varepsilon$, $t_n \in (t^-(x_n), t^+(x_n))$ such that $v_n \to v$. Calling $\mathcal{O}_v := \{\eta(t, v) \mid t \in \mathbf{R}\}$ the orbit of $\eta(\cdot, v)$, it is easy to see that, passing to the limit, $\mathcal{O}_v \cap \bar{B}_\varepsilon \neq \emptyset$; namely, there exist $x \in \bar{B}_\varepsilon$ and $\tau \in \mathbf{R}$ such that $v = \eta(\tau, x)$.

Then, either $\Phi(1, v) = c^\pm$ and case (i) or (ii) holds; or $\Phi(1, v) \in (c^-, c^+)$. In this latter case, $\tau \in (t^-(x), t^+(x))$. Hence $x \in \partial B_\varepsilon$, for otherwise, $v \in Q$. Furthermore, $\Phi(1, \eta(s, x)) \in (c^-, c^+)$ for s near τ. It is clear that $\eta(s, x) \in \partial Q$ for s near τ. Indeed, $\eta(s, x) = \lim_n \eta(s, x_n)$, $x_n \in B_\varepsilon$, $x_n \to x$, and therefore $s \in (t^-(x), t^+(x))$.

The proof of Proposition 1.28 is complete. ■

Conclusion of Case (b). Define the mountain pass level

$$c(\lambda) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\lambda, \gamma(t)),$$

where

$$\Gamma := \left\{ \gamma \in C([0, 1], \bar{Q}) \mid \gamma(0) = 0 \text{ and } \gamma(1) \in T^- \right\} \tag{1.60}$$

and

$$T^- := \left\{ v \in S^- \ : \ \Phi(1, v) = c^- \right\} \neq \emptyset .$$

For $\lambda > 1$, as in (1.55) we conclude that $c(\lambda) \geq \alpha(\lambda) > 0$.

We can now obtain the existence of a Palais–Smale sequence $v_n \in Q$ at the level $c(\lambda)$, i.e., such that

$$\Phi(\lambda, v_n) \to c(\lambda), \quad \nabla\Phi(\lambda, v_n) \to 0 . \tag{1.61}$$

The argument (see for the proof Remark B.4) is based on a standard deformation argument using that $-\nabla\Phi(1, v)$ is a pseudogradient vector field for $\Phi(\lambda, v)$ for all $v \in \partial Q$ and constructing, thanks to Proposition 1.28(iii), a pseudogradient vector field that leaves Q invariant under its flow.

By compactness, $v_n \to \bar{v} \in \bar{Q}$. Actually, $\bar{v} \in \operatorname{int} Q$ because $\nabla\Phi(\lambda, v) \neq 0$ $\forall v \in \partial Q$ for λ sufficiently close to 1. Again $\bar{v} \neq 0$ because $\Phi(\lambda, \bar{v}) = c(\lambda) > 0$ and $\Phi(\lambda, 0) = 0$.

The proof of the Fadell–Rabinowitz theorem, Theorem 1.21, is complete. ■

Remark 1.34. So far, periodic solutions close to the equilibrium were obtained from assumptions on the linearized vector field only. The periodic solutions found are the continuations of the linear normal modes having periods close to the periods of the linearized equation. If, in contrast, the Hamiltonian vector field is "sufficiently smooth and nonlinear," then an abundance of periodic solutions with large period is expected (Birkhoff–Lewis orbits [34]); see also [23]. An extension of these results to Hamiltonian PDEs has been given in [14], [31].

Before concluding this chapter we remark that global variational methods for finding periodic solutions of Hamiltonian systems as critical points of the action functional have been successfully used since the pioneering papers by Rabinowitz [117] and Weinstein [130], where, independently, the existence of at least one periodic solution was established on each compact and strictly convex energy hypersurface. We refer to the books by Ekeland [56], Mawhin–Willem [91], and Hofer–Zehnder [72].

We also quote the longstanding conjecture stating the existence of at least n geometrically distinct periodic orbits (closed characteristics) on *any* compact and convex energy hypersurface (this is a nonperturbative analogue of the Weinstein–Moser theorem).

Ekeland and Hofer [57] have proved the existence of two closed characteristics (Theorem 1.20 can be seen as a particular case of this result).

The best results available at the moment are by Long and Zhu [86], where the existence of at least $[n/2] + 1$ closed characteristics was proved.

We shall not, however, pursue this path, but we shall describe the extensions of the Lyapunov, Weinstein–Moser, and Fadell–Rabinowitz theorems regarding bifurcation of small-amplitude periodic solutions of Hamiltonian PDEs.

2

Infinite Dimension

We want to extend the local bifurcation theory of periodic solutions described in the previous chapter to infinite-dimensional Hamiltonian PDEs.

Let us consider the autonomous nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + a_1(x)u = a_2(x)u^2 + a_3(x)u^3 + \cdots \,, \\ u(t,0) = u(t,\pi) = 0 \,, \end{cases} \tag{2.1}$$

which possesses the equilibrium solution $u \equiv 0$.

As in the previous chapter we pose the following question.

- QUESTION: do there exist periodic solutions of (2.1) close to $u = 0$?

The first step is again to study the linearized equation

$$\begin{cases} u_{tt} - u_{xx} + a_1(x)u = 0 \,, \\ u(t,0) = u(t,\pi) = 0 \,. \end{cases} \tag{2.2}$$

The self-adjoint Sturm–Liouville operator $-\partial_{xx} + a_1(x)$ possesses a basis $\{\varphi_j\}_{j\geq 1}$ of eigenvectors with real eigenvalues λ_j,

$$(-\partial_{xx} + a_1(x))\varphi_j = \lambda_j \varphi_j \,, \qquad \lambda_j \to +\infty \,. \tag{2.3}$$

The φ_j are orthonormal with respect to the L^2 scalar product.

In this basis, equation (2.2) reduces to infinitely many decoupled linear oscillators: $u(t,x) = \sum_j u_j(t)\varphi_j(x)$ is a solution of (2.2) if and only if

$$\ddot{u}_j + \lambda_j u_j = 0 \qquad j = 1, 2, \ldots \,. \tag{2.4}$$

If $-\partial_{xx} + a_1(x)$ is positive definite, all its eigenvalues $\lambda_j > 0$ are positive and $u = 0$ looks like an "infinite-dimensional elliptic equilibrium" for (2.2) with linear frequencies of oscillations

$$\omega_j := \sqrt{\lambda_j} \,;$$

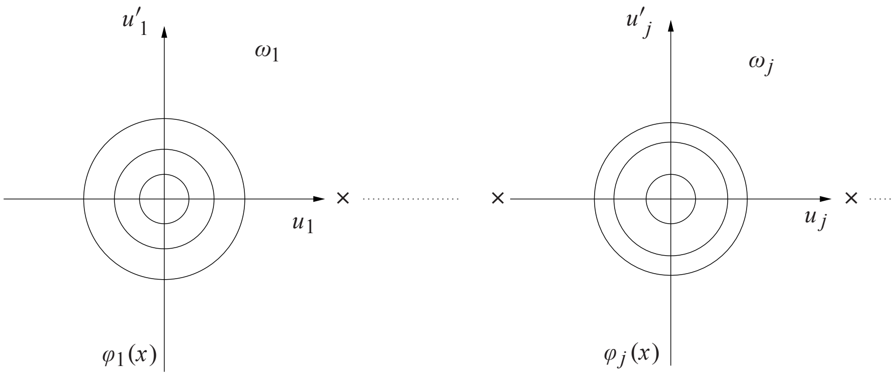

Fig. 2.1. Equation (2.2) is equivalent, in the basis of eigenvectors φ_j, to infinitely many decoupled harmonic oscillators with frequencies ω_j.

see Figure 2.1. The quadratic Hamiltonian that generates (2.2),

$$H_2(u, p) = \int_0^\pi \frac{p^2}{2} + \frac{u_x^2}{2} + a_1(x)\frac{u^2}{2}\,\mathrm{d}x\,,$$

where $p := u_t$, is positive definite and, in coordinates, can be written

$$H_2 = \sum_{j\geq 1} \frac{p_j^2 + \lambda_j u_j^2}{2}\,,$$

where $p_j := \dot{u}_j \in l^2$ (Plancherel theorem).

Remark 2.1. If some $\lambda_j < 0$ (there are at most finitely many negative eigenvalues since $\lambda_j \to +\infty$) then the corresponding linear equation (2.4) describes an harmonic repulsor (hyperbolic directions).

The general solution of (2.2) is given by the linear superposition of infinitely many oscillations of amplitude a_j, frequency ω_j, and phase θ_j on the normal modes φ_j:

$$u(t, x) = \sum_{j\geq 1} a_j \cos(\omega_j t + \theta_j)\varphi_j(x)\,.$$

Hence all solutions of (2.2) are either *periodic* in time, *quasiperiodic*, or *almost periodic*.

A solution u is periodic when each of the frequencies ω_j for which the amplitude a_j is nonzero (active frequencies) is an integer multiple of a basic frequency ω_0:

$$\omega_j = l_j\omega_0\,, \qquad l_j \in \mathbf{Z}\,.$$

In this case u is $2\pi/\omega_0$-periodic in time.

The solution u is quasiperiodic with an m-dimensional frequency base if there is an m-dimensional frequency vector $\omega_0 \in \mathbf{R}^m$ with rationally independent components (i.e., $\omega_0 \cdot k \neq 0, \forall k \in \mathbf{Z}^m \setminus \{0\}$) such that the active frequencies satisfy

$$\omega_j = l_j \cdot \omega_0 \,, \qquad l_j \in \mathbf{Z}^m \,.$$

A solution is called almost periodic otherwise, namely, if there is not a finite number of base frequencies.

It is a natural question to ask whether some of these periodic, quasiperiodic, or almost-periodic solutions of the linear equation (2.2) persist in the nonlinear equation (2.1).

2.1 The Lyapunov Center Theorem for PDEs

Bifurcation of small-amplitude periodic and quasiperiodic solutions of (2.1) was indeed obtained by Kuksin [81] and Wayne [128], extending KAM theory. Next, Craig–Wayne [51] introduced the Lyapunov–Schmidt reduction method for Hamiltonian PDEs to handle the nonlinear wave equation (2.1) with periodic spatial boundary conditions where some resonance phenomena appear due to the near coincidence of pairs of linear frequencies.

We start by describing the Craig–Wayne result [51], which is an extension of the Lyapunov center theorem to the nonlinear wave equation (2.1).

The main difficulty to overcome is the appearance of a

(i) *"small divisors" problem*

(which in finite dimensions arises only in the search for quasiperiodic solutions).

To explain how it arises, recall the key nonresonance hypothesis (1.9) in the Lyapunov center theorem,[1]

$$\omega_j - l\omega_1 \neq 0 \,, \quad \forall l \in \mathbf{Z} \,, \quad \forall j = 2, \ldots, n \,.$$

In finite dimensions, for *any* ω sufficiently close to ω_1, the same condition $\omega_j - l\omega \neq 0$, $\forall l \in \mathbf{Z}$, $\forall j = 2, \ldots, n$, holds, and the standard implicit function theorem can be applied.

In contrast, the eigenvalues of the Sturm–Liouville problem (2.3) grow polynomially[2] like $\lambda_j \approx j^2 + O(1)$ for $j \to +\infty$ (as is seen by lower and upper comparison with the operator with constant coefficients; see e.g., [111]), and therefore $\omega_j = j + o(1)$. As a consequence, in infinite dimension, when $\omega_1 \notin \mathbf{Q}$, the set

$$\{\omega_j - l\omega_1 \,, \ \forall l \in \mathbf{Z} \,, \ j = 2, 3, \ldots\}$$

[1] The eigenvalues of the finite-dimensional matrix A of the previous chapter correspond here to $\pm \mathrm{i}\omega_j$, $j = 1, \ldots$.

[2] For example, the eigenvalues of $-\partial_{xx} + m$ are $\lambda_j = j^2 + m$ with eigenvectors $\sin(jx)$.

accumulates to zero, and the nonresonance condition

$$\omega_j - l\omega_1 \neq 0\,,\ \forall l \in \mathbf{Z}\,,\ j = 2, 3, \dots\,, \tag{2.5}$$

is not sufficient for application of the standard implicit function theorem.

This is the “small divisors” problem (this name is due to the fact that such quantities appear as denominators), which shall be described in more detail at the end of this chapter and in Chapter 4.

Nevertheless, replacing (2.5) with some stronger condition can ensure the persistence of a large Cantor-like set of small-amplitude periodic solutions of (2.1).

A typical result is the following.

Theorem 2.2. (Craig–Wayne [51]) *Let*

$$f(x, u) := a_1(x)u - a_2(x)u^2 - a_3(x)u^3 + \cdots$$

be a function analytic in the region $\{(x, u) \mid |\mathrm{Im}\, x| < \sigma\,, |u| < 1\}$ *and odd* $f(-x, -u) = -f(x, u)$. *Among this class of nonlinearities there is an open dense set* $\mathcal{F}$ *(in the* C^0 *topology) such that* $\forall f \in \mathcal{F}$, *there exist* $r_* > 0$, *a Cantor-like set* $\mathcal{C} \subset [0, r_*)$ *of positive measure, and a* C^∞ *function* $\Omega(r)$ *with* $\Omega(0) = \omega_1$ *such that* $\forall r \in \mathcal{C}$, *there exists a periodic solution* $u(t, x; r)$ *of (2.1) with frequency* $\Omega(r)$. *These solutions are analytic in* (x, t) *and satisfy*

$$|u(t, x; r) - r\cos(\Omega(r)t)\varphi_1(x)| \leq Cr^2\,, \qquad |\Omega(r) - \omega_1| < Cr^2\,.$$

The Lyapunov solutions $u(t, x; r)$ are parametrized with the amplitude r, but also the corresponding set of frequencies $\Omega(r)$, $r \in \mathcal{C}$, has positive measure.

The conditions on the terms $a_1(x)$, $a_2(x)$, $a_3(x)$, etc. are, roughly, the following: first a condition on $a_1(x)$ to avoid primary resonances on the linear frequencies ω_j (which depend on a_1); see the nonresonance condition (2.5). Next, a condition of genuine nonlinearity placed on $a_2(x)$, $a_3(x)$ is required to solve the 2-dimensional bifurcation equation.

Some cases of “partially resonant” PDEs, where the bifurcation equation is $2m$-dimensional, have been next studied in [52].

The “completely resonant” case

$$a_1(x) \equiv 0\,,$$

where

$$\omega_j = j\,, \qquad \forall j \in \mathbf{N} \tag{2.6}$$

(infinitely many resonance relations among the linear frequencies) remained an open problem.

In this case *all* the solutions of (2.2) are 2π-periodic. This is the analogous situation considered in finite dimensions by the Weinstein–Moser and Fadell–Rabinowitz resonant center theorems.

For infinite-dimensional Hamiltonian PDEs, aside from the small-divisor problem (i), this leads to the further complication of an *infinite-dimensional bifurcation phenomenon*.

In the sequel we shall discuss the extension to Hamiltonian PDEs of the results of Weinstein–Moser and Fadell–Rabinowitz. The required infinite-dimensional resonance variational analysis is the main contribution of [24]–[25] and [27].

Remark 2.3. To prove existence of quasiperiodic solutions with m-frequencies

$$u(t,x) = U(\omega t, x)\,, \quad \omega \in \mathbf{R}^m\,,$$

where $U(\cdot, x)\colon \mathbf{T}^m \to \mathbf{R}$, the main difficulty with respect to the periodic case lies in a more complicated geometry of the numbers $\omega \cdot l - \omega_j$, $l \in \mathbf{Z}^m$, $j \in \mathbf{N}$. Existence of quasiperiodic solutions with the Lyapunov–Schmidt approach has been proved by Bourgain [35], [36]. For existence results via the KAM approach, see, e.g., [109], [45], [84], [82], and references therein.

2.2 Completely Resonant Wave Equations

From now on we shall consider the completely resonant PDE

$$\begin{cases} u_{tt} - u_{xx} = f(x,u)\,, \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{2.7}$$

where

$$f(x,u) = a_p(x)u^p + O(u^{p+1})\,, \qquad p \geq 2\,.$$

The linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0\,, \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{2.8}$$

possesses an infinite-dimensional linear space of periodic solutions with the same period 2π, called the kernel of the D'Alembert operator

$$\Box := \partial_{tt} - \partial_{xx}$$

with Dirichlet boundary conditions. Indeed, any solution of (2.8) can be written as

$$v(t,x) = \sum_{j \geq 1} a_j \cos(jt + \theta_j)\sin(jx)$$

(*Fourier method*), and it is 2π-periodic in time.

It will be also convenient to express the solutions of the wave equation (2.8) using the *D'Alembert method*, namely, representing

$$v(t,x) = \eta(t+x) - \eta(t-x)\,, \tag{2.9}$$

where η is any 2π-periodic function, as the linear superposition of two waves with the opposite profile η traveling in opposite directions.

Lemma 2.4. *Every solution of (2.8) can be written in the form (2.9).*

Proof. The general solution of $\Box u = 0$ can be written as

$$u(t,x) = p(t+x) + q(t-x)$$

for any $p, q\colon \mathbf{R} \to \mathbf{R}$. Indeed, in the "characteristic" variables $s_+ := t + x$, $s_- := t - x$, the equation

$$(\partial_{tt} - \partial_{xx})u = (\partial_t + \partial_x) \circ (\partial_t - \partial_x)u = 0$$

reads $\partial_{s^+} \circ \partial_{s^-} u = 0$. Imposing the Dirichlet boundary conditions

$$\begin{cases} u(t,0) = p(t) + q(t) = 0\,, & \forall t\,, \\ u(t,\pi) = p(t+\pi) + q(t-\pi) = 0\,, & \forall t\,, \end{cases}$$

we get

$$q(t) = -p(t) \quad \text{and} \quad p(t+\pi) = -q(t-\pi) = p(t-\pi)\,, \quad \forall t\,.$$

This last relation shows that $p(\cdot)$ is a 2π-periodic function, and we have written $u(t,x) = p(t+x) - p(t-x)$ in the form (2.9). ∎

For proving existence of small-amplitude periodic solutions of (2.7), the new difficulty to overcome is the following:

(ii) the presence of an *infinite*-dimensional space of periodic solutions of the linear equation (2.8) with the same period: which solutions of (2.8) can be continued to solutions of the nonlinear equation (2.7)?

The first existence results for completely resonant wave equations were obtained by Lidskij–Shulman [87] and Bambusi–Paleari [15] for $f(u) = u^3$. The small-divisor problem (i) is bypassed, imposing on the frequency ω a strong nonresonance condition (see (2.16) below) that is satisfied on a zero-measure set $\mathcal{W}_\gamma$. The choice of the nonlinearity $f(u) = u^3$ is due to the method used to solve the bifurcation equation.

In [24], solving the *infinite*-dimensional bifurcation equation via min–max variational methods, we proved existence of periodic solutions of (2.7) for the same zero-measure set of frequencies $\mathcal{W}_\gamma$, but for a *general* nonlinearity. For simplicity we now present these results for the nonlinearities

$$f(u) := au^p\,, \qquad a \neq 0\,, \qquad p \geq 2\,,$$

where p is either an *odd* or an *even* integer.

We start with the easier case that p is odd.

2.3 The Case p Odd

Since the nonlinear wave equation (2.7) is *autonomous*, the period of the solutions is a priori unknown, and we introduce it as a free parameter.

Normalizing the period, we look for 2π-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} = a u^p \, , \\ u(t,0) = u(t,\pi) = 0 \, , \end{cases} \tag{2.10}$$

in the Banach space

$$X := \Big\{ u \in H^1(\mathbf{\Omega}, \mathbf{R}) \cap L^\infty(\mathbf{\Omega}, \mathbf{R}) \mid u(t,0) = u(t,\pi) = 0, \ u(-t,x) = u(t,x) \Big\} \, ,$$

where $\mathbf{\Omega} := (\mathbf{R}/2\pi\mathbf{Z}) \times [0,\pi]$, endowed with norm

$$\|u\| := \|u\|_\infty + \|u\|_{H^1}$$

(note that $H^1(\mathbf{\Omega})$ does not embed into $L^\infty(\mathbf{\Omega})$ since $\mathbf{\Omega}$ is a bidimensional domain).

The space X is an algebra with respect to multiplication of functions, i.e., there exists $C > 0$ such that

$$\|u_1 u_2\| \le C \|u_1\| \, \|u_2\| \, , \qquad \forall u_1, u_2 \in X \, .$$

We can look for solutions even in time because equation (2.7) is reversible.

We shall make assumptions on the frequency ω in (2.16).

Remark 2.5. Any $u \in X$ can be developed in a Fourier series as

$$u(t,x) = \sum_{l \ge 0, j \ge 1} u_{l,j} \cos lt \sin jx \, ,$$

and its H^1-norm and scalar product are

$$\|u\|_{H^1}^2 := \int_\Omega u_t^2 + u_x^2 = \frac{\pi^2}{2} \sum_{l \ge 0, j \ge 1} u_{l,j}^2 (j^2 + l^2) \, , \tag{2.11}$$

$$(u,w)_{H^1} := (u,w) := \int_\Omega u_t w_t + u_x w_x = \frac{\pi^2}{2} \sum_{l \ge 0, j \ge 1} u_{l,j} w_{l,j} (l^2 + j^2)$$

$\forall u, w \in X$. We define

$$\|u\|_{L^2} := \Big(\int_\Omega |u|^2 \Big)^{1/2} \qquad \text{and} \qquad (u,w)_{L^2} := \int_\Omega u \, w \, ,$$

respectively the L^2-norm and L^2-scalar product.

2.3.1 The Variational Lyapunov–Schmidt Reduction

Instead of looking for solutions of (2.10) taking values in a shrinking neighborhood of 0, it is convenient to perform the rescaling

$$u \to \delta u \,, \qquad \delta > 0 \,,$$

obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} = \varepsilon a u^p \,, \\ u(t,0) = u(t,\pi) = 0 \,, \end{cases} \tag{2.12}$$

where

$$\varepsilon := \delta^{p-1} \,.$$

Equation (2.12) is the Euler–Lagrange equation of the *Lagrangian action functional* $\Psi \in C^1(X, \mathbf{R})$ defined by

$$\Psi(u) := \Psi_{\omega,\varepsilon}(u) := \int_0^{2\pi} \mathrm{d}t \int_0^{\pi} \Big[\frac{\omega^2}{2} u_t^2 - \frac{1}{2} u_x^2 + \varepsilon F(u)\Big] \,\mathrm{d}x \,, \tag{2.13}$$

where

$$F(u) := \int_0^u f(s)\,\mathrm{d}s = a \frac{u^{p+1}}{p+1} \,.$$

To find critical points of Ψ, we perform a Lyapunov–Schmidt reduction, decomposing

$$X = V \oplus W \,,$$

where

$$\begin{aligned} V &:= \Big\{ v = \sum_{j \geq 1} a_j \cos(jt) \sin jx \;\Big|\; a_j \in \mathbf{R},\ \sum_{j \geq 1} j^2 a_j^2 < +\infty \Big\} \\ &= \Big\{ v = \eta(t+x) - \eta(t-x) \;\Big|\; \eta(\cdot) \in H^1(\mathbf{T}),\ \eta \text{ odd} \Big\} \end{aligned} \tag{2.14}$$

are the solutions in X of the linear equation (2.8), and

$$\begin{aligned} W &:= \Big\{ w \in X \;\Big|\; (w, v)_{L^2} = 0,\ \forall\, v \in V \Big\} \\ &= \Big\{ \sum_{l \geq 0, j \geq 1} w_{l,j} \cos(lt) \sin jx \in X \;\Big|\; w_{j,j} = 0 \;\forall j \geq 1 \Big\} . \end{aligned}$$

W is also the H^1-orthogonal of V in X.

Remark 2.6. On V the norm $\| \ \|$ is equivalent to the H^1-norm $\| \ \|_{H^1}$, since

$$\|v\|_\infty \leq C \|v\|_{H^1} \,, \qquad \forall v \in V \,.$$

Moreover, recalling (2.14), the embedding

$$(V, \|\cdot\|_{H^1}) \hookrightarrow (V, \|\cdot\|_\infty) \tag{2.15}$$

is compact because $H^1(\mathbf{T}) \hookrightarrow L^\infty(\mathbf{T})$ is compact.

Projecting (2.12), for $u := v + w$ with $v \in V$, $w \in W$, yields

$$\begin{cases} \omega^2 v_{tt} - v_{xx} = \varepsilon \Pi_V f(v+w) & \textit{bifurcation equation}\,, \\ \omega^2 w_{tt} - w_{xx} = \varepsilon \Pi_W f(v+w) & \textit{range equation}\,, \end{cases}$$

where $\Pi_V : X \to V$ and $\Pi_W : X \to W$ are the (continuous) projectors respectively on V and W.

The existence of a solution $w := w(\varepsilon, \omega, v)$ of the range equation cannot be derived for any $\omega \approx 1$, $\varepsilon \approx 0$ (as in finite dimensions) by the standard implicit function theorem at $\omega = 1$, $\varepsilon = 0$. Indeed, even though $(\partial_{tt} - \partial_{xx})^{-1} : W \to W$ is compact, it gains just one derivative (see Lemma 5.4) but the term $(\omega^2 - 1)\partial_{tt}$ loses two derivatives.

2.3.2 The Range Equation

We first solve the range equation assuming that the frequency ω satisfies the following nonresonance condition:

$$\omega \in \mathcal{W}_\gamma := \left\{ \omega \in \mathbf{R} \;\middle|\; |\omega l - j| \geq \frac{\gamma}{l}, \quad \forall (l, j) \in \mathbf{N} \times \mathbf{N}\,,\ j \neq l \right\} \tag{2.16}$$

for some $\gamma > 0$, introduced in [15].

For example, by Liouville's theorem, Theorem D.6, a quadratic irrational ω (namely an irrational root of a second-degree polynomial with integer coefficients) belongs to $\mathcal{W}_\gamma$ for some $\gamma := \gamma(\omega) > 0$.

Note that by Dirichlet's theorem, Theorem D.1, the condition in (2.16) is the strongest possible requirement of this kind: if $\tau < 1$, there are no real numbers ω such that $|\omega l - j| \geq (\gamma / l^\tau)$, $\forall l \neq j$. This follows also from (D.5).

Moreover, still by Dirichlet's theorem (or (D.5)), the set $\mathcal{W}_\gamma$ is empty if $\gamma \geq 1$.

Remark 2.7. Actually, $\mathcal{W}_\gamma$ is empty if $\gamma \geq 1/\sqrt{5}$, because, by Hurwitz's theorem (see Theorem 2F in [121]), if x is irrational, there exist infinitely many distinct rational numbers p/q such that $|x - p/q| < 1/\sqrt{5}\, q^2$.

Lemma 2.8. *For $0 < \gamma \leq 1/4$ the set $\mathcal{W}_\gamma$ is uncountable, has zero measure, and accumulates to $\omega = 1$ both from the left and from the right.*

Proof. We claim that if $\gamma \in (0, 1/4]$, the set $\mathcal{W}_\gamma$ contains uncountably many irrational numbers ω such that its continued fraction expansion is

$$\omega = [1, a_1, a_2, \ldots] := 1 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

for every $a_1 \in \mathbf{N}$ and

$$a_i \in \{1, 2\}\,, \quad \forall i \geq 2 \tag{2.17}$$

(we refer to Appendix D for the basic notions about continued fractions). All these numbers ω belong to the open neighborhood $(1, 1 + a_1^{-1})$, and for $a_1 \to +\infty$, they accumulate to $\omega = 1$.

Let us prove the previous claim. Take any pairs of integers $(l, j) \in \mathbf{N} \times \mathbf{N}$, $l \neq j$. If

$$|\omega l - j| \geq \frac{1}{2l},$$

then the condition in the definition of $\mathcal{W}_\gamma$, $|\omega l - j| \geq \gamma / l$, is obviously satisfied because $\gamma \leq 1/4$. On the other hand, if

$$|\omega l - j| < \frac{1}{2l},$$

then, by a theorem of Legendre (see Theorem D.8), j/l coincides with one of the convergents of ω,

$$1, \ [1, a_1], [1, a_1, a_2], \ldots, [1, a_1, a_2, \ldots, a_n], \ldots;$$

see Definition D.7.

The quotient j/l is different from the first convergent of ω, i.e., $j/l \neq 1$, because $l \neq j$ by the definition of $\mathcal{W}_\gamma$. Therefore

$$\frac{j}{l} = [1, a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} \qquad \text{for some} \quad n \geq 1,$$

and since q_n and p_n are relatively prime by (D.4), q_n divides l and $l \geq q_n$. As a consequence, in order to get $|\omega l - j| \geq \gamma / l$ with $\gamma \in (0, 1/4]$, it is enough to prove that

$$|\omega q_n - p_n| \geq \frac{1}{4q_n}, \qquad \forall n \geq 1. \tag{2.18}$$

Every convergent p_n/q_n of ω satisfies, see (D.6),

$$\left|\omega - \frac{p_n}{q_n}\right| \geq \frac{1}{q_n^2(a_{n+1} + 2)}, \tag{2.19}$$

whence, since by (2.17), $a_{n+1} \in \{1, 2\}$, $\forall n \geq 1$, we have

$$\left|\omega - \frac{p_n}{q_n}\right| \geq \frac{1}{q_n^2(2 + 2)} = \frac{1}{4q_n^2},$$

proving (2.18).

In conclusion, we have proved that

$$\forall (l, j) \in \mathbf{N} \times \mathbf{N}, l \neq j, \ |\omega l - j| \geq \frac{\gamma}{l} \quad \text{with} \quad \gamma \in (0, 1/4],$$

namely that $\omega \in \mathcal{W}_\gamma$.

Remark 2.9. If in (2.17) we take $a_i \in \{1, 2, \dots, N\}$, $\forall i \geq 2$, we deduce by (2.19) that $\omega \in \mathcal{W}_\gamma$ for all $\gamma \in (0, 1/(2+N)]$.

Similarly, the uncountably many irrational numbers

$$\omega := [0, 1, a_2, a_3, a_4, \dots]$$

for every $a_2 \in \mathbf{N}$ and with $a_3, a_4, \dots \in \{1, 2\}$ belong to the set

$$\mathcal{W}_\gamma \cap \Big(1 - \frac{1}{a_2 + 1}, 1\Big) \quad \text{with} \quad \gamma \in (0, 1/4] \; .$$

For $a_2 \to +\infty$ these numbers accumulate to $\omega = 1$ from the left. ■

2.1. Exercise: If $\omega \in \mathcal{W}_\gamma$, $\omega > 1$, then also $2 - \omega < 1$ belongs to $\mathcal{W}_\gamma$. This gives an alternative proof that $\mathcal{W}_\gamma$ accumulates to $\omega = 1$ also from the left.

Lemma 2.10. *Let $\omega \in \mathcal{W}_\gamma \cap (1/2, 3/2)$ for some $\gamma > 0$. Then the operator*

$$L_\omega := \omega^2 \partial_{tt} - \partial_{xx}$$

restricted to W has a bounded inverse defined by

$$L_\omega^{-1} w := \sum_{j \geq 1, l \geq 0, l \neq j} \frac{w_{l,j}}{-\omega^2 l^2 + j^2} \cos(lt) \sin(jx) \, , \quad \forall w \in W \, , \tag{2.20}$$

which satisfies

$$\|L_\omega^{-1} w\| \leq \frac{C}{\gamma} \|w\| \tag{2.21}$$

for a positive constant C independent of γ and ω.

Proof. The eigenvalues

$$\Big\{ -\omega^2 l^2 + j^2 \mid l \geq 0, \; j \geq 1, \; l \neq j \Big\}$$

of $(L_\omega)_{|W}$ satisfy, for $\omega \in \mathcal{W}_\gamma$,

$$|-\omega^2 l^2 + j^2| = |\omega l - j|(\omega l + j) \geq \frac{\gamma}{l}(\omega l + j) \geq \gamma \omega \geq \frac{\gamma}{2} \quad \forall l \neq j \, , \; l \geq 1 \, , \tag{2.22}$$

for $\omega > 1/2$ (for $l = 0$ the inequality (2.22) is trivial). Therefore, by (2.20) and (2.11),

$$\|L_\omega^{-1} w\|_{H^1} \leq \frac{C}{\gamma} \|w\|_{H^1} \, . \tag{2.23}$$

We claim also that

$$\|L_\omega^{-1} w\|_\infty \leq \frac{C}{\gamma} \|w\|_{H^1} \, . \tag{2.24}$$

The inequalities (2.23)–(2.24) imply (2.21).

The key observation to prove (2.24) is that the lower bound (2.22) can be significantly improved if j is not the closest integer $e(l)$ to ωl, defined by

$$|e(l) - \omega l| = \min_{j \in \mathbf{N}} |j - \omega l| \, .$$

By (2.20),

$$\|L_\omega^{-1} w\|_\infty \leq \sum_{l \geq 0, j \geq 1, j \neq l} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)} = S_1 + S_2 \, ,$$

where

$$S_1 := \sum_{l \geq 0, j \geq 1, j \neq l, j \neq e(l)} \frac{|w_{l,j}|}{|\omega l - j|(\omega l + j)}$$

and

$$S_2 := \sum_{l \geq 0, e(l) \neq l} \frac{|w_{l,e(l)}|}{|\omega l - e(l)|(\omega l + e(l))} \, .$$

Estimate of S_1. For $j \neq e(l)$ we have

$$|j - \omega l| \geq |j - e(l)| - |e(l) - \omega l| \geq |j - e(l)| - \frac{1}{2} \geq \frac{|j - e(l)|}{2} \, .$$

Moreover, since $|e(l) - \omega l| < 1/2$, and $\omega \geq 1/2$, we have $4\omega l \geq e(l) + l$, and hence

$$4(j + \omega l) \geq j + e(l) + l \geq |j - e(l)| + l \, .$$

Defining $w_{l,j}$ by $w_{l,j} = 0$ if $j \leq 0$ or $j = l$, we get

$$S_1 \leq \sum_{l \geq 0, j \in \mathbf{Z}, j \neq e(l)} \frac{8|w_{l,j}|}{|j - e(l)|(|j - e(l)| + l)} \leq 8 R_1 \|w\|_{L^2}$$

by the Cauchy–Schwarz inequality, where

$$R_1^2 := \sum_{l \geq 0, j \in \mathbf{Z}, j \neq e(l)} \frac{1}{(j - e(l))^2(|j - e(l)| + l)^2} = \sum_{l \geq 0, j \in \mathbf{Z}, j \neq 0} \frac{1}{j^2(|j| + l)^2}$$
$$\leq \sum_{l \geq 0, j \in \mathbf{Z}, j \neq 0} \frac{1}{j^2(1 + l)^2} < \infty \, .$$

Estimate of S_2. Using (2.22),

$$S_2 \leq \frac{2}{\gamma} \sum_{e(l) \neq l} |w_{l,e(l)}| \leq \frac{C}{\gamma} \|w\|_{H^1}$$

by the Cauchy–Schwarz inequality. ■

Fixed points of the nonlinear operator $\mathcal{G}: W \to W$ defined by

$$\mathcal{G}(\varepsilon, \omega; w) := \varepsilon L_\omega^{-1} \Pi_W f(v + w)$$

are solutions of the range equation.

Lemma 2.11. (Solution of the range equation) *Assume* $\omega \in \mathcal{W}_\gamma$. $\forall R > 0$, $\exists$ $\varepsilon_0(R) > 0$, $C_0(R) > 0$ *such that* $\forall v \in B_{2R} := \{v \in V \mid \|v\|_{H^1} \le 2R\}$, $\forall 0 \le \varepsilon\gamma^{-1} \le \varepsilon_0(R)$ *there exists a unique solution* $w(\varepsilon, v) \in W$ *of the range equation satisfying*

$$\|w(\varepsilon, v)\| \le C_0(R)\frac{\varepsilon}{\gamma}. \tag{2.25}$$

Moreover, the map $v \to w(\varepsilon, v)$ *is in* $C^1(B_{2R}, W)$.

Proof. By the Banach algebra property of X and (2.21),

$$\|\mathcal{G}(\varepsilon, \omega; w)\| \le \varepsilon\frac{C}{\gamma}\|v + w\|^p \le \varepsilon\frac{C'}{\gamma}((2R)^p + \|w\|^p), \quad \forall v \in B_{2R}.$$

Hence, for $\rho := 2(C'/\gamma)\varepsilon(2R)^p$ and $\forall 0 \le \varepsilon\gamma^{-1} \le \varepsilon_0(R)$ so small that $\rho \le 2R$, $\mathcal{G}(\varepsilon, \omega; B_\rho) \subset B_\rho$, where $B_\rho := \{w \in W \mid \|w\| \le \rho\}$. Similarly we can prove

$$\|D\mathcal{G}(\varepsilon, \omega; w)\| \le \frac{1}{2}, \quad \forall v \in B_{2R},$$

$\forall 0 \le \varepsilon\gamma^{-1} \le \varepsilon_0(R)$ small enough. We conclude by the contraction mapping theorem. The regularity of $w(\varepsilon, v)$ with respect to v follows by the implicit function theorem. ■

2.3.3 The Bifurcation Equation

It remains to solve the infinite-dimensional bifurcation equation

$$\omega^2 v_{tt} - v_{xx} = \varepsilon \Pi_V f(v + w(\varepsilon, v)). \tag{2.26}$$

Equation (2.26) is the Euler–Lagrange equation of the *reduced Lagrangian action functional* $\Phi_\varepsilon: B_{2R} \to \mathbf{R}$ defined by

$$\Phi_\varepsilon(v) := \Psi(v + w(\varepsilon, v)),$$

where Ψ is the Lagrangian action functional introduced in (2.13).

Indeed, w solves the range equation if and only if it is a critical point of the restricted functional $w \to \Psi(v + w) \in \mathbf{R}$, namely if and only if

$$\begin{aligned} D\Psi(v + w)[h] &= \int_\Omega \omega^2(v_t + w_t)h_t - (v_x + w_x)h_x + \varepsilon f(v + w)h \\ &= \int_\Omega \omega^2 w_t h_t - w_x h_x + \varepsilon f(v + w)h = 0, \ \forall h \in W \end{aligned} \tag{2.27}$$

because $\int_\Omega v_t h_t = \int_\Omega v_x h_x = 0$. Therefore, $\forall h \in V$,

$$\begin{aligned} D\Phi_\varepsilon(v)[h] &= D\Psi(v + w(\varepsilon, v))\Big[h + Dw(\varepsilon, v)[h]\Big] \\ &= D\Psi(v + w(\varepsilon, v))[h] \\ &= \int_\Omega \omega^2 v_t h_t - v_x h_x + \varepsilon \Pi_V f(v + w(\varepsilon, v))h \end{aligned} \tag{2.28}$$

because $Dw(\varepsilon, v)[h] \in W$, (2.27), and $\int_\Omega w_t h_t = \int_\Omega w_x h_x = 0$, $\forall h \in V$; see Figure 2.2.

Remark 2.12. By (2.28), since $w(\varepsilon, \cdot) \in C^1(B_{2R}, W)$, $D\Phi_\varepsilon \in C^1(B_{2R}, V)$ and therefore $\Phi_\varepsilon \in C^2(B_{2R}, \mathbf{R})$.

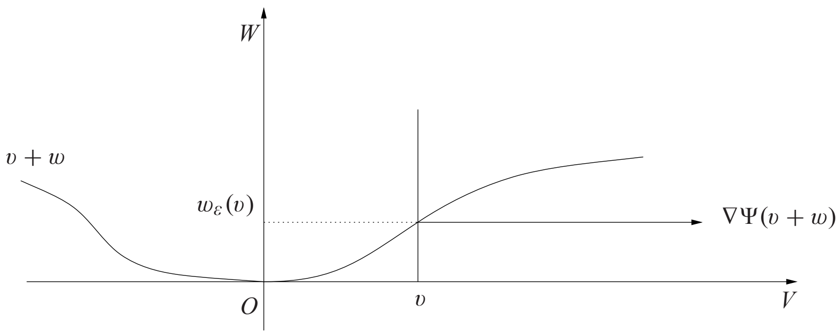

Fig. 2.2. The variational reduction: by (2.27) the gradient $\nabla\Psi(v + w(\varepsilon, v))$ is parallel to V.

To find critical points of Φ_ε we expand

$$\begin{aligned} \Phi_\varepsilon(v) &= \int_\Omega \frac{\omega^2}{2}(v_t + (w(\varepsilon, v))_t)^2 - \frac{1}{2}(v_x + (w(\varepsilon, v))_x)^2 + \varepsilon F(v + w(\varepsilon, v)) \\ &= \int_\Omega \frac{\omega^2}{2} v_t^2 - \frac{v_x^2}{2} + \varepsilon F(v + w(\varepsilon, v)) - \frac{\varepsilon}{2} f(v + w(\varepsilon, v)) w(\varepsilon, v)\,, \end{aligned}$$

because by orthogonality,

$$\int_\Omega v_t w_t = \int_\Omega v_x w_x = 0\,,$$

and, inserting $h = w(\varepsilon, v) \in W$ in (2.27),

$$\int_\Omega \omega^2 w_t^2 - w_x^2 + \varepsilon f(v + w(\varepsilon, v)) w(\varepsilon, v) = 0\,.$$

Hence, using that $\|v_t\|_{L^2}^2 = \|v_x\|_{L^2}^2 = \|v\|_{H^1}^2/2$,

$$\Phi_\varepsilon(v) = \frac{\omega^2 - 1}{4}\|v\|_{H^1}^2 + \varepsilon \int_\Omega \Big[F(v + w(\varepsilon, v)) - \frac{1}{2} f(v + w(\varepsilon, v)) w(\varepsilon, v) \Big] . \quad (2.29)$$

To find nontrivial critical points of (2.29) we have to impose a suitable relation between the frequency ω and the amplitude ε (ω must tend to 1 as $\varepsilon \to 0$). We set the "frequency–amplitude" relation

$$\frac{\omega^2 - 1}{2} = -\varepsilon \,\mathrm{sign}(a)$$

(recall $f(u) := au^p$). By (2.29),

$$\Phi_\varepsilon(v) = -\varepsilon \,\mathrm{sign}(a) \Big[\frac{1}{2}\|v\|_{H^1}^2 - |a| \int_\Omega \frac{v^{p+1}}{p+1} + \mathcal{R}_\varepsilon(v) \Big] ,$$

where

$$\mathcal{R}_\varepsilon(v) := -|a| \int_\Omega \Big\{ \frac{1}{p+1} \Big[(v + w(\varepsilon, v))^{p+1} - v^{p+1} \Big] - \frac{1}{2}(v + w(\varepsilon, v))^p w(\varepsilon, v) \Big\}$$

satisfies, using (2.25),

$$|\mathcal{R}_\varepsilon(v)| , |(\nabla \mathcal{R}_\varepsilon(v), v)| \le C_1(R) \frac{\varepsilon}{\gamma} \quad (2.30)$$

for some constant $C_1(R)$. Furthermore, by Remark 2.12, $\mathcal{R}_\varepsilon \in C^2(B_{2R}, \mathbf{R})$.

The functional (which we still denote by Φ_ε)

$$\Phi_\varepsilon(v) := \frac{1}{2}\|v\|_{H^1}^2 - |a| \int_\Omega \frac{v^{p+1}}{p+1} + \mathcal{R}_\varepsilon(v) \quad (2.31)$$

possesses a local minimum at the origin, and one could think to prove the existence of nontrivial critical points via the mountain pass theorem of Ambrosetti and Rabinowitz [7] (Theorem B.8 in the appendix), see Figure 2.3. Note that since p is *odd*, the functional $\int_\Omega v^{p+1}$ is strictly positive, $\forall v \neq 0$.

However, we cannot apply Theorem B.8 directly, since Φ_ε is defined only in a neighborhood of the origin.

2.3.4 The Mountain Pass Argument

Step 1: *Extension of* Φ_ε.
We define the extended action functional $\widetilde{\Phi}_\varepsilon \in C^2(V, \mathbf{R})$ as

$$\widetilde{\Phi}_\varepsilon(v) := \frac{\|v\|_{H^1}^2}{2} - |a| \int_\Omega \frac{v^{p+1}}{p+1} + \widetilde{\mathcal{R}}_\varepsilon(v) ,$$

where $\widetilde{\mathcal{R}}_\varepsilon \colon V \to \mathbf{R}$ is

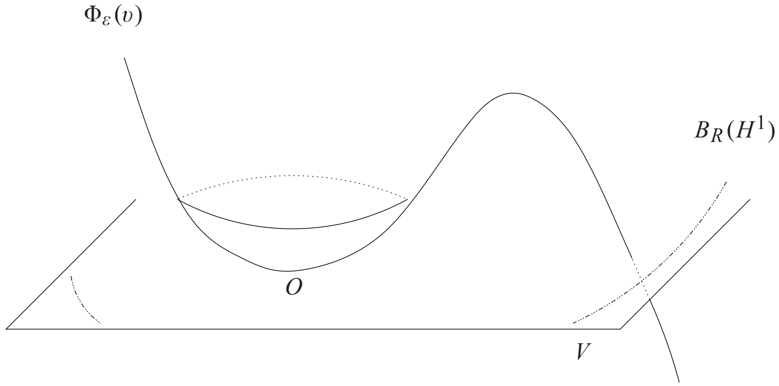

Fig. 2.3. The mountain pass geometry of Φ_ε.

$$\widetilde{\mathcal{R}}_\varepsilon(v) := \lambda\Big(\frac{\|v\|_{H^1}^2}{R^2}\Big)\mathcal{R}_\varepsilon(v)$$

and $\lambda\colon [0,+\infty) \to [0,1]$ is a smooth, nonincreasing cutoff function such that

$$\lambda(x) = \begin{cases} 1 & \text{for } |x| \le 1\,, \\ 0 & \text{for } |x| \ge 4\,, \end{cases}$$

and $|\lambda'(x)| < 1$. By definition,

$$\widetilde{\Phi}_\varepsilon(v) = \begin{cases} \Phi_\varepsilon(v) & \text{for } \|v\|_{H^1} \le R\,, \\ \dfrac{\|v\|_{H^1}^2}{2} - |a|\displaystyle\int_\Omega \frac{v^{p+1}}{p+1} & \text{for } \|v\|_{H^1} \ge 2R\,. \end{cases} \tag{2.32}$$

Furthermore, by (2.30) and the definition of λ,

$$|\widetilde{\mathcal{R}}_\varepsilon(v)|\,, |(\nabla\widetilde{\mathcal{R}}_\varepsilon(v), v)| \le C_1(R)\frac{\varepsilon}{\gamma}\,. \tag{2.33}$$

Step 2: $\widetilde{\Phi}_\varepsilon$ *satisfies the geometrical hypotheses of Theorem B.8.*
Let $0 < \rho < R$. For all $\|v\|_{H^1} = \rho$ we have

$$\begin{aligned} \widetilde{\Phi}_\varepsilon(v) &\overset{(2.32)}{=} \Phi_\varepsilon(v) \\ &\overset{(2.31)}{=} \frac{\|v\|_{H^1}^2}{2} - |a|\int_\Omega \frac{v^{p+1}}{p+1} + \mathcal{R}_\varepsilon(v) \\ &\overset{(2.30)}{\ge} \frac{1}{2}\rho^2 - \kappa_1\rho^{p+1} - \frac{\varepsilon}{\gamma}C_1(R)\,. \end{aligned} \tag{2.34}$$

Fix $\rho > 0$ such that $(\rho^2/2) - \kappa_1\rho^{p+1} \ge \rho^2/4$. By (2.34), we get for

$$0 < \frac{\varepsilon}{\gamma} C_1(R) \le \frac{\rho^2}{8} \tag{2.35}$$

that

$$\widetilde{\Phi}_\varepsilon(v) \ge \frac{1}{8}\rho^2 > 0 \quad \text{if} \quad \|v\|_{H^1} = \rho\,, \tag{2.36}$$

verifying the assumption (*ii*) of Theorem B.8.

Let us verify assumption (*iii*). Recalling (2.32), for every $\|v_0\|_{H^1} = 1$, there exists $\widetilde{t}$ large enough that

$$\widetilde{\Phi}_\varepsilon(\widetilde{t} v_0) = \frac{\widetilde{t}^2}{2} - |a| \frac{\widetilde{t}^{p+1}}{p+1} \int_\Omega {v_0}^{p+1} < 0 \tag{2.37}$$

because $p \ge 3$ is an *odd* integer.

Remark 2.13. In contrast, if p is even, then $\int_\Omega v^{p+1} = 0$, $\forall v \in V$, see (2.47), and another development is necessary; see Section 2.4.

We define the minimax class

$$\Gamma := \left\{ \gamma \in C([0,1]\,, V) \mid \gamma(0) = 0\,, \gamma(1) = \widetilde{v} \right\},$$

where $\widetilde{v} := \widetilde{t} v_0$ (so that $\widetilde{\Phi}_\varepsilon(\widetilde{v}) < 0$ by (2.37)), and the mountain pass level

$$c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \widetilde{\Phi}_\varepsilon(\gamma(s))\,. \tag{2.38}$$

Since any path $\gamma \in \Gamma$ intersects the sphere $\{v \in V \mid \|v\|_{H^1} = \rho\}$, by (2.36) we get

$$c_\varepsilon \ge \min_{\|v\|_{H^1} = \rho} \widetilde{\Phi}_\varepsilon(v) \ge \frac{1}{8}\rho^2 > 0\,. \tag{2.39}$$

By Theorem B.8 we deduce the existence of a Palais–Smale sequence for $\widetilde{\Phi}_\varepsilon$ at the level $c_\varepsilon > 0$, namely, there exists a sequence $v_n \in V$ such that

$$\widetilde{\Phi}_\varepsilon(v_n) \to c_\varepsilon\,, \ \nabla \widetilde{\Phi}_\varepsilon(v_n) \to 0 \tag{2.40}$$

(see Definition B.5).

Our aim is to prove that the Palais–Smale sequence v_n converges, up to a subsequence, to some nontrivial critical point $v_* \ne 0$ in some open ball of V where $\widetilde{\Phi}_\varepsilon$ and Φ_ε coincide.

We need some bounds for the H^1-norm of v_n independent of ε.

Step 3: *Confinement of the Palais–Smale sequence.*

We first get a bound for the mountain-pass level c_ε independent of ε. By the definition (2.38) and (2.33),

$$c_\varepsilon \le \max_{s \in [0,1]} \widetilde{\Phi}_\varepsilon(s\widetilde{v}) \le \max_{s \in [0,1]} \left[\frac{s^2}{2} \|\widetilde{v}\|_{H^1}^2 - |a| \frac{s^{p+1}}{p+1} \int_\Omega \widetilde{v}^{p+1} \right] + 1 =: \kappa \tag{2.41}$$

for $0 < C_1(R)\gamma^{-1}\varepsilon < 1$. Then

$$\begin{aligned}\widetilde{\Phi}_\varepsilon(v_n) - \frac{(\nabla\widetilde{\Phi}_\varepsilon(v_n), v_n)}{p+1} &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|v_n\|_{H^1}^2 + \widetilde{\mathcal{R}}_\varepsilon(v_n) - \frac{(\nabla\widetilde{\mathcal{R}}_\varepsilon(v_n), v_n)}{p+1} \\ &\overset{(2.33)}{\geq} \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|v_n\|_{H^1}^2 - 2C_1(R)\gamma^{-1}\varepsilon \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|v_n\|_{H^1}^2 - 1 \end{aligned} \tag{2.42}$$

for

$$0 < 2C_1(R)\frac{\varepsilon}{\gamma} \leq 1\,. \tag{2.43}$$

By (2.40), for n large,

$$\widetilde{\Phi}_\varepsilon(v_n) - \frac{(\nabla\widetilde{\Phi}_\varepsilon(v_n), v_n)}{p+1} \leq (c_\varepsilon + 1) + \|v_n\|_{H^1} \overset{(2.41)}{\leq} \kappa + 1 + \|v_n\|_{H^1}\,,$$

and by (2.42), we derive

$$\kappa + 1 + \|v_n\|_{H^1} \geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|v_n\|_{H^1}^2 - 1\,,$$

whence

$$\|v_n\|_{H^1} \leq R_* \tag{2.44}$$

for a suitable constant R_* independent of ε. The estimate (2.44) holds for $0 < \varepsilon\gamma^{-1} < C_*(R)$ for some $C_*(R) > 0$ (recall (2.35) and (2.43)).

Step 4: *Existence of a nontrivial critical point.*

Let us fix

$$\bar{R} := R_* + 1 \quad \text{and take} \quad 0 < \varepsilon\gamma^{-1} \leq C_*(\bar{R})\,.$$

By (2.44), definitively for n large, $v_n \in B_{\bar{R}}$, and so

$$\widetilde{\Phi}_\varepsilon(v_n) = \Phi_\varepsilon(v_n)\,. \tag{2.45}$$

Hence

$$\nabla\widetilde{\Phi}_\varepsilon(v_n) = \nabla\Phi_\varepsilon(v_n) = v_n - \nabla G(v_n) + \nabla\mathcal{R}_\varepsilon(v_n) \to 0\,, \tag{2.46}$$

where we have set

$$G(v) := \frac{|a|}{p+1}\int_\Omega v^{p+1}\,.$$

Lemma 2.14. (Compactness) *$\nabla G\colon V \to V$ and $\nabla\mathcal{R}_\varepsilon\colon B_{\bar{R}} \to V$ are compact operators.*

Proof. Let $\|v_k\|_{H^1}$ be bounded. Then, up to a subsequence, $v_k \rightharpoonup \overline{v} \in V$ weakly in H^1 and $v_k \to \overline{v}$ in $\|\cdot\|_\infty$ by the compact embedding (2.15). Moreover, since also $w_k := w(\varepsilon, v_k)$ is bounded, we can assume that $w_k \rightharpoonup \overline{w}$ weakly in H^1 and $w_k \to \overline{w}$ in the $\|\cdot\|_{L^q}$ norm for all $q < \infty$. Since

$$(\nabla G(v), h) = \int_\Omega |a| v^p h\,,$$

it follows easily that $\nabla G(v_k) \to \nabla G(\overline{v})$ in V. We claim that $\nabla \mathcal{R}_\varepsilon(v_k) \to \bar{\mathcal{R}}$, where

$$(\bar{\mathcal{R}}, h) = \int_\Omega |a|(\overline{v} + \overline{w})^p h - |a|\overline{v}^p h = (\nabla \mathcal{R}_\varepsilon(\overline{v}), h)\,.$$

Indeed, since $w_k \to \overline{w}$ in $\|\cdot\|_{L^q}$, it converges (up to a subsequence) also almost everywhere. We can deduce, by the Lebesgue dominated convergence theorem, that $(v_k + w_k)^p \to (\overline{v} + \overline{w})^p$ in L^2 since $(v_k + w_k)^p \to (\overline{v} + \overline{w})^p$ almost everywhere and $(v_k + w_k)^p$ is bounded in L^∞. Hence, since

$$(\nabla \mathcal{R}_\varepsilon(v_k), h) = \int_\Omega |a|(v_k + w_k)^p h - |a| v_k^p h\,,$$

$\nabla \mathcal{R}_\varepsilon(v_k) \to \bar{\mathcal{R}}$. ■

Since ∇G and $\nabla \mathcal{R}_\varepsilon$ are compact and the Palais–Smale sequence v_n is bounded in H^1, by Lemma B.7 in the appendix, v_n is precompact and therefore converges to a nontrivial critical point v_ε of Φ_ε.

Remark 2.15. As $\varepsilon \to 0$, v_ε converges to the mountain pass critical set of

$$\frac{1}{2}\|v\|_{H^1}^2 - \frac{|a|}{(p+1)}\int_\Omega v^{p+1}\,;$$

see [25], [27].

In conclusion, we have found a nontrivial solution

$$u = \delta\Big(v_\varepsilon + w(\varepsilon, v_\varepsilon)\Big) \in X$$

of (2.10).

We have finally proved the following theorem.

Theorem 2.16. ([24]) (Existence, p odd) *Let $f(u) = au^p$ ($a \neq 0$) for an odd integer $p \geq 3$. There exists $C_2(f) > 0$ such that $\forall \omega \in \mathcal{W}_\gamma$ satisfying $|\omega - 1|\gamma^{-1} \leq C_2(f)$ and $\omega < 1$ if $a > 0$ (resp. $\omega > 1$ if $a < 0$), equation (2.7) possesses at least one nontrivial small-amplitude $2\pi/\omega$-periodic solution $u_\omega \neq 0$. The solution u_ω converges to zero like $\|u_\omega\| \leq C|\omega - 1|^{1/(p-1)}$ as $\omega \to 1$. See Figure 2.4.*

Remark 2.17. **(Regularity)** By a bootstrap argument, the periodic solution u_ω of Theorem 2.16 is in C^2; see [25]. Actually, we could also prove that u_ω is analytic; see the next chapters.

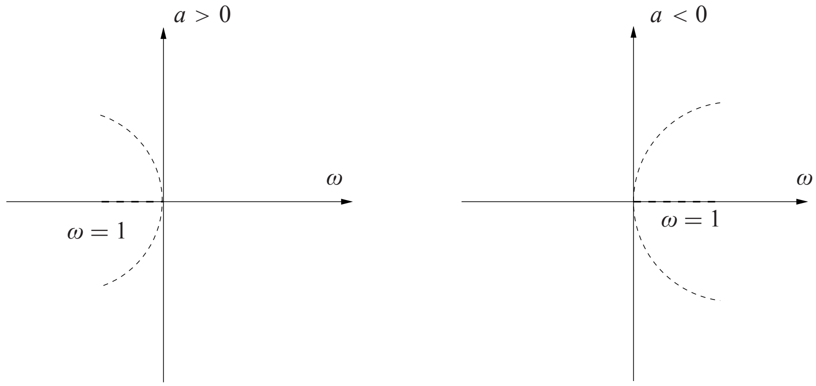

Fig. 2.4. The bifurcation diagram of Theorem 2.16. In general, u_ω does not vary smoothly with the frequency ω. Since equation (2.7) is autonomous, there is a whole circle of periodic solutions $u_\omega(\cdot + \theta, x)$, $\forall \theta \in \mathbf{R}$.

2.4 The Case p Even

The case $f(u) = au^p$ with p an even integer is more difficult because

$$\int_\Omega v^{p+1} \equiv 0\,, \qquad \forall v \in V\,, \tag{2.47}$$

and so the development (2.31) no longer implies the mountain pass geometry of Φ_ε. The identity (2.47) follows by a more general lemma:

Lemma 2.18. *Let* $v_1, \ldots, v_{2k+1} \in V$. *Then*

$$\int_\Omega v_1 \cdots v_{2k+1} = 0\,. \tag{2.48}$$

In particular, if $v \in V$ *then* $v^{2k} \in W$.

Proof. Write $v_j = \eta_j(t+x) - \eta_j(t-x) \in V$ in the D'Alembertian form (2.9). By the change of variables $(t, x) \mapsto (t, \pi - x)$ and periodicity,

$$\begin{aligned}
\int_\Omega v_1 \cdots v_{2k+1} &= \int_0^\pi \int_{\mathbf{T}} \prod_{j=1}^{2k+1} \Big(\eta_j(t+x) - \eta_j(t-x)\Big)\, \mathrm{d}t\, \mathrm{d}x \\
&= \int_0^\pi \int_{\mathbf{T}} \prod_{j=1}^{2k+1} \Big(\eta_j(t+\pi-x) - \eta_j(t-\pi+x)\Big)\, \mathrm{d}t\, \mathrm{d}x \\
&= \int_0^\pi \int_{\mathbf{T}} \prod_{j=1}^{2k+1} \Big(\eta_j(t-x) - \eta_j(t+x)\Big)\, \mathrm{d}t\, \mathrm{d}x \\
&= (-1)^{2k+1} \int_\Omega v_1 \cdots v_{2k+1} = -\int_\Omega v_1 \cdots v_{2k+1}\,,
\end{aligned}$$

which implies (2.48). ■

To find critical points of Φ_ε we have to develop at higher orders in ε the non-quadratic term in (2.29). Using (2.47) we obtain

$$\int_\Omega F(v + w(\varepsilon, v)) - \frac{1}{2} f(v + w(\varepsilon, v)) w(\varepsilon, v) = \frac{a}{2} \int_\Omega v^p w(\varepsilon, v) + \mathcal{R}_1 , \tag{2.49}$$

where, setting $w_\varepsilon := w(\varepsilon, v)$,

$$\begin{aligned} \mathcal{R}_1 &:= \frac{a}{p+1} \int_\Omega \Big[(v + w_\varepsilon)^{p+1} - v^{p+1} - (p+1) v^p w_\varepsilon \Big] \\ &\quad - \frac{a}{2} \int_\Omega \big[(v + w_\varepsilon)^p - v^p \big] w_\varepsilon \\ &= O(\|w_\varepsilon\|^2) \overset{(2.25)}{=} O(\varepsilon^2 \gamma^{-2}) . \end{aligned}$$

Substitute into (2.49) the expression

$$\begin{aligned} w(\varepsilon, v) &= \varepsilon a L_\omega^{-1} \Pi_W v^p + \varepsilon a L_\omega^{-1} \Pi_W \Big[(v + w(\varepsilon, v))^p - v^p \Big] \\ &= \varepsilon a L_\omega^{-1} v^p + \mathcal{R}_2 , \end{aligned}$$

where

$$\mathcal{R}_2 := \varepsilon a L_\omega^{-1} \Pi_W \Big[(v + w(\varepsilon, v))^p - v^p \Big] \overset{(2.21)}{=} O(\varepsilon \gamma^{-1} \|w_\varepsilon\|) \overset{(2.25)}{=} O(\varepsilon^2 \gamma^{-2}) .$$

We get

$$\int_\Omega F(v + w(\varepsilon, v)) - \frac{1}{2} f(v + w(\varepsilon, v)) w(\varepsilon, v) = \varepsilon \frac{a^2}{2} \int_\Omega v^p L_\omega^{-1} v^p + \mathcal{R}_3 ,$$

where

$$\mathcal{R}_3 := \frac{a}{2} \int_\Omega v^p \mathcal{R}_2 + \mathcal{R}_1 = O(\varepsilon^2 \gamma^{-2}) , \tag{2.50}$$

and, by (2.29),

$$\begin{aligned} \Phi_\varepsilon(v) &= \frac{\omega^2 - 1}{4} \|v\|_{H^1}^2 + \varepsilon^2 \frac{a^2}{2} \int_\Omega v^p L_\omega^{-1} v^p + \varepsilon \mathcal{R}_3 \\ &= -\varepsilon^2 \Big[\frac{1}{2} \|v\|_{H^1}^2 - \frac{a^2}{2} \int_\Omega v^p L_\omega^{-1} v^p - \varepsilon^{-1} \mathcal{R}_3 \Big] , \end{aligned}$$

having set the frequency–amplitude relation

$$\frac{\omega^2 - 1}{2} = -\varepsilon^2 . \tag{2.51}$$

Still denoting, for notational convenience, $-\varepsilon^{-2} \Phi_\varepsilon$ by the same symbol Φ_ε, we are reduced to finding critical points of

$$\Phi_\varepsilon(v) = \frac{1}{2}\|v\|_{H^1}^2 - \frac{a^2}{2}\int_\Omega v^p L_\omega^{-1} v^p - \varepsilon^{-1}\mathcal{R}_3$$
$$= \frac{1}{2}\|v\|_{H^1}^2 - \frac{a^2}{2}\int_\Omega v^p \Box^{-1} v^p + \mathcal{R}_\varepsilon\,, \tag{2.52}$$

where $\Box^{-1}: W \to W$ is the inverse of the D'Alembert operator $\Box := L_1 := \partial_{tt} - \partial_{xx}$, and the remainder

$$\mathcal{R}_\varepsilon := -\varepsilon^{-1}\mathcal{R}_3 - \frac{a^2}{2}\int_\Omega v^p (L_\omega^{-1} - \Box^{-1}) v^p \tag{2.53}$$

satisfies

$$\mathcal{R}_\varepsilon(v)\,, (\nabla \mathcal{R}_\varepsilon(v), v) = O(\varepsilon\gamma^{-2})\,. \tag{2.54}$$

Indeed, by (2.50) the term $\varepsilon^{-1}\mathcal{R}_3$ satisfies (2.54). The second term in (2.53) satisfies an even better estimate. Develop in Fourier series

$$v^p =: r = \sum_{l \neq j} r_{l,j} \cos(lt)\sin(jx)$$

(recall that $v^p \in W$ by Lemma 2.18). Hence

$$\Box^{-1} v^p = \sum_{j \neq l} \frac{r_{l,j}}{-l^2 + j^2}\cos(lt)\sin(jx)\,,\ L_\omega^{-1} v^p = \sum_{j \neq l} \frac{r_{l,j}}{-\omega^2 l^2 + j^2}\cos(lt)\sin(jx)$$

and

$$\Big|\int_\Omega v^p (L_\omega^{-1} - \Box^{-1}) v^p\Big| = c\Big|\sum_{j \neq l} \frac{(\omega^2 - 1) r_{l,j}^2 l^2}{(\omega^2 l^2 - j^2)(l^2 - j^2)}\Big|$$
$$\leq C\frac{\varepsilon^2}{\gamma}\sum_{j \neq l} r_{l,j}^2\, l^2 \leq C'\frac{\varepsilon^2}{\gamma}\|r\|_{H^1}^2 \leq C'\frac{\varepsilon^2}{\gamma}\|v^p\|^2$$

using (2.51), $|\omega^2 l^2 - j^2| \geq \gamma/2$, $|l^2 - j^2| \geq 1$. The estimate for $(\nabla \mathcal{R}_\varepsilon(v), v)$ can be proved similarly.

Now, to show that Φ_ε in (2.52) has the mountain pass geometry it remains only to check that $\int_\Omega v^p \Box^{-1} v^p > 0$ for some $v \in V$. This is proved with the method of characteristics.

Lemma 2.19. $\int_\Omega v^p \Box^{-1} v^p > 0,\ \forall v \neq 0.$

Proof. Write $v = \eta(t + x) - \eta(t - x) \in V$ and

$$v^p = m(t + x, t - x) \quad \text{with} \quad m(s_1, s_2) = \Big(\eta(s_1) - \eta(s_2)\Big)^p\,.$$

Let $z := \Box^{-1}(v^p)$ be the solution of

$$\begin{cases} z_{tt} - z_{xx} = v^p\,, \\ z(t,0) = z(t,\pi) = 0\,, \\ z(t+2\pi, x) = z(t,x)\,. \end{cases}$$

Making the change of variables $(s_1, s_2) = (t + x, t - x)$, such an equation with the boundary conditions becomes, for $z(t,x) = M(t+x, t-x)$,

$$\begin{cases} \partial_{s_2 s_1} M(s_1, s_2) = (1/4) m(s_1, s_2)\,, \\ M(s_1, s_1) = M(s_1, s_1 - 2\pi) = 0\,, \\ M(s_1 + 2\pi, s_2 + 2\pi) = M(s_1, s_2) \quad \forall s_2 \le s_1 \le s_2 + 2\pi\,. \end{cases}$$

We claim that a solution of this problem is

$$M(s_1, s_2) := \frac{1}{8} \int_{Q_{s_1, s_2}} m(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\,, \tag{2.55}$$

where

$$Q_{s_1,s_2} := \Big\{ (\xi_1, \xi_2) \in \mathbf{R}^2 \mid s_1 \le \xi_1 \le s_2 + 2\pi,\ s_2 \le \xi_2 \le s_1 \Big\}\,.$$

Indeed, differentiating with respect to s_1 yields

$$\partial_{s_1} M(s_1, s_2) = \frac{1}{8} \int_{s_1}^{s_2 + 2\pi} m(\xi_1, s_1)\, \mathrm{d}\xi_1 - \frac{1}{8} \int_{s_2}^{s_1} m(s_1, \xi_2)\, \mathrm{d}\xi_2.$$

Differentiating with respect to s_2, since $m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^p$ with p even and η is 2π-periodic, gives us

$$\partial_{s_2} \partial_{s_1} M = \frac{1}{8} m(s_2 + 2\pi, s_1) + \frac{1}{8} m(s_1, s_2) = \frac{1}{4} m(s_1, s_2).$$

From (2.55), $M \ge 0$, and, by Lemma 2.20 below,

$$\int_\Omega v^p \Box^{-1} v^p\, \mathrm{d}t\, \mathrm{d}x = \frac{1}{2} \int_{\mathbf{T}^2} m(s_1, s_2) M(s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2 \ge 0\,.$$

If $\int_\Omega v^p \Box^{-1} v^p = 0$ then $m(s_1, s_2) = 0$ or $M(s_1, s_2) = 0$. If $M(s_1, s_2) = 0$ then $m(\xi_1, \xi_2) = 0$ for all $(\xi_1, \xi_2) \in Q_{s_1, s_2}$, and so $m(s_1, s_2) = 0$. In both cases $m(s_1, s_2) = 0$, and so $v = 0$. ■

Lemma 2.20. *Let* $m \colon \mathbf{T}^2 \to \mathbf{R}$. *Then*

$$\int_\Omega m(t + x, t - x)\, \mathrm{d}t\, \mathrm{d}x = \frac{1}{2} \int_{\mathbf{T}^2} m(s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2. \tag{2.56}$$

Proof. Make the change of variables $(s_1, s_2) = (t + x, t - x)$ and use the periodicity of m. ■

With the same mountain-pass-type argument developed in Section 2.3 we can prove the following existence theorem:

Theorem 2.21. [24] (Existence, p even) *Let $f(u) = au^p$ ($a \neq 0$) for an even integer p. There exists $C_3(f) > 0$ such that $\forall \omega \in \mathcal{W}_\gamma$, $\omega < 1$, with $|\omega - 1|\gamma^{-4} \leq C_3(f)$, equation (2.7) possesses at least one nontrivial, small-amplitude $2\pi/\omega$-periodic solution $u_\omega \neq 0$. The solution u_ω converges to zero like $\|u_\omega\| \leq C|\omega - 1|^{1/2(p-1)}$ for $\omega \to 1$. See Figure 2.5.*

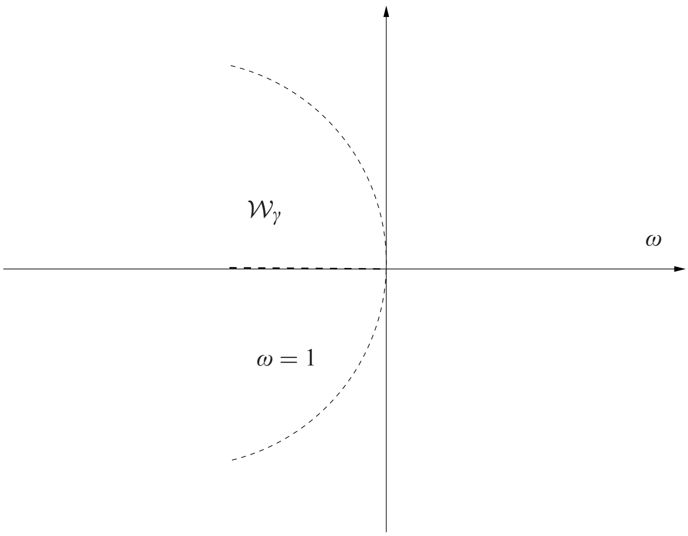

Fig. 2.5. The Cantor families of solutions for p even. There exist small-amplitude nontrivial periodic solutions just for $\omega < 1$. Also in this case, in general, u_ω does not depend smoothly on ω.

Remark 2.22. This greater difficulty for finding periodic solutions when the nonlinearity f is nonmonotone is a common feature for nonlinear wave equations; see, e.g., [114], [42], [43], [47]. In physical terms there is no "confinement effect" due to the potential. Actually, in [29]–[30] a nonexistence result is proved for even-power nonlinearities in the case of spatial periodic boundary conditions. This highlights that the term $\int_\Omega v^p \Box^{-1} v^p$, which is responsible for the existence result of Theorem 2.21, is a "boundary effect" due to the Dirichlet conditions.

Remark 2.23. In [29]–[30], the existence of quasiperiodic solutions with two frequencies for completely resonant nonlinear wave equations with spatial periodic boundary condition has been obtained. After a similar variational Lyapunov–Schmidt reduction the reduced-action functional turns out to have the infinite-dimensional linking geometry [19].

2.5 Multiplicity

As in the Weinstein–Moser and Fadell–Rabinowitz resonant center theorems, one could expect multiplicity of periodic solutions. This could be proved by exploiting the S^1 symmetry of the action functional induced by time translations. However, in this case we can provide a much more detailed picture of these solutions.

Critical points of the reduced-action functional Φ_ε restricted to each subspace $V_n \subset V$ formed by the functions

$$v = \eta(n(t+x)) - \eta(n(t-x)) \in V\,, \quad n \in \mathbf{N}\,,$$

which are $2\pi/n$-periodic, are critical points of Φ_ε on the whole of V.

As in the previous section we can prove that for each $n \geq 1$, the reduced-action functional $\Phi_{\varepsilon|V_n}$ possesses a mountain pass critical point v_n.

It turns out that the larger n is, the smaller $|\omega - 1|$ has to be (i.e., ε). In this way, for $\omega \in \mathcal{W}_\gamma$ sufficiently close to 1, we can prove the existence of a *large* number N_ω of $2\pi/\omega$-periodic solutions

$$u_1, \ldots, u_n, \ldots, u_{N_\omega}$$

of (2.7), where $N_\omega \to +\infty$ as $\omega \to 1$. Making refined estimates we can obtain

$$N_\omega \approx \sqrt{\frac{\gamma^\tau}{|\omega - 1|}}$$

for some $\tau \in [1, 2]$; see [25].

Remark 2.24. An estimate like $N_\omega \approx 1/\sqrt{|\omega - 1|}$ is, in general, the optimal one. Indeed, the eigenvalues of $(L_\omega)_{|V}$ are $-\omega^2 l^2 + j^2 = (-\omega^2 + 1)j^2$ since $l^2 = j^2$. Hence, to find solutions of (2.7), if $N^2|\omega^2 - 1| = O(1)$, we can perform a finite-dimensional Lyapunov–Schmidt reduction to the N-dimensional subspace $V_N := \{v = \sum_{j=1}^N a_j \cos(jt)\sin(jx)\} \subset V$. We expect N mountain pass critical points of the corresponding reduced-action functional on V_N.

The periodic solutions u_n have increasing energies (in n) and rotate with greater and greater frequency: actually in [25] it is proved that the *minimal period* of the nth solution u_n is $2\pi/n\omega$. This, in particular, proves that the solutions u_n are geometrically distinct.

Theorem 2.25. ([25]) (Multiplicity, p odd) *Let $f(u) = au^p$ ($a \neq 0$) for an odd integer $p \geq 3$. Then there exists a positive constant $C_4 := C_4(f)$ such that $\forall\omega \in \mathcal{W}_\gamma$ and $\forall n \in \mathbf{N}\backslash\{0\}$ satisfying*

$$\frac{|\omega - 1|n^2}{\gamma} \leq C_4 \tag{2.57}$$

and $\omega < 1$ if $a > 0$ (resp. $\omega > 1$ if $a < 0$), equation (2.7) possesses at least one periodic classical C^2 solution with minimal period $2\pi/(n\omega)$.

As a consequence, there exist $u_1, \dots, u_n, \dots, u_{N_\omega}$, *with*[3]

$$N_\omega := \left[\frac{\gamma C_4}{|\omega - 1|} \right] - 1$$

geometrically distinct periodic solutions of (2.7) with the same period $2\pi/\omega$.

Theorem 2.26. ([25]) (Multiplicity, p even) *Let* $f(u) = au^p$ $(a \neq 0)$ *for an even integer* p. *Then there exists a positive constant* C_5 *depending only on* f *such that* $\forall \omega \in \mathcal{W}_\gamma$ *with* $\omega < 1$, $\forall n \geq 2$ *such that*

$$\frac{(|\omega - 1| n^2)^{1/2}}{\gamma} \leq C_5 \,, \tag{2.58}$$

equation (2.7) possesses at least one periodic classical C^2 *solution with minimal period* $2\pi/(n\omega)$.

As a consequence, there exist $u_2, \dots, u_n, \dots, u_{N_\omega}$, *with*

$$N_\omega := \left[\frac{(\gamma C_5)^2}{|\omega - 1|} \right] - 1$$

geometrically distinct periodic solutions of (2.7) with the same period $2\pi/\omega$. *If* $p = 2$, *the existence result holds true for* $n = 1$ *as well.*

2.6 The Small-Divisor Problem

We now want to discuss the "small divisors" problem (i) to prove existence of periodic solutions for a set of *positive* (actually asymptotically full) measure of frequencies ω.

For this we have to relax the nonresonance condition in (2.16) requiring

$$\omega \in \mathcal{W}_{\gamma,\tau} := \left\{ \omega \in \mathbf{R} \mid |\omega l - j| \geq \frac{\gamma}{l^\tau} \,, \quad \forall (l, j) \in \mathbf{N} \times \mathbf{N} \,, \ j \neq l \right\} \tag{2.59}$$

for some $\tau > 1$, $\gamma > 0$ (note that $\tau = 1$ in (2.16)).

By standard measure estimates the set $\mathcal{W}_{\gamma,\tau}$ has positive measure; it actually has asymptotically full density as $\omega \to 1$. The proof is instructive.

Proposition 2.27. *Let* $\tau > 1$, $\gamma \in (0, 1/3)$. *Then*

$$\lim_{\eta \to 0} \frac{|\mathcal{W}_{\gamma,\tau} \cap (1, 1 + \eta)|}{|\eta|} = 1 \,. \tag{2.60}$$

---

[3] $[x] \in \mathbf{Z}$ denotes the integer part of $x \in \mathbf{R}$.

Proof. To fix the ideas suppose $\eta > 0$. First note that $\forall \omega \in (1, 1+\eta)$, if $l \neq j$ and $1 \leq l \leq (2/3\eta)$, then

$$|\omega l - j| = |(l-j) + (\omega - 1)l| \geq 1 - |\eta| l \geq \frac{1}{3}$$

and the condition in (2.59) is trivially satisfied ($\gamma < 1/3$).

We have to estimate the measure of the complementary set

$$\mathcal{W}^c_{\gamma,\tau} \cap (1, 1+\eta) = \bigcup_{(j,l)\in S} R_{jl}\,, \tag{2.61}$$

where

$$R_{jl} := \left(\frac{j}{l} - \frac{\gamma}{l^{\tau+1}}, \frac{j}{l} + \frac{\gamma}{l^{\tau+1}}\right) \cap (1, 1+\eta)$$

and

$$S := \left\{(j,l) \in \mathbf{N}\times\mathbf{N}\,,\ j \neq l\,,\ l > \frac{2}{3\eta}\right\}.$$

Note that if $(j,l) \in S$ does not satisfy

$$(1-4\eta)l < j < (1+4\eta)l\,,$$

then for η small enough,

$$R_{jl} = \emptyset\,.$$

So we can restrict in (2.61) to the subset of indexes

$$\widetilde{S} := \left\{(j,l) \in \mathbf{N}\times\mathbf{N}\,,\ j \neq l\,,\ l > \frac{2}{3\eta}\,,\ (1-4\eta)l < j < (1+4\eta)l\right\},$$

obtaining the estimate

$$\begin{aligned}
|\mathcal{W}^c_{\gamma,\tau} \cap (1, 1+\eta)| &\leq \sum_{(j,l)\in\widetilde{S}} |R_{jl}| \\
&\leq \sum_{l\geq 2/3\eta,\ (1-4\eta)l<j<(1+4\eta)l} \ \sum \frac{2\gamma}{l^{\tau+1}} \\
&\leq \sum_{l\geq 2/3\eta} 8\eta l\,\frac{2\gamma}{l^{\tau+1}} = 16\eta\gamma \sum_{l\geq 2/3\eta} \frac{1}{l^{\tau}} \\
&= O(\gamma\,\eta^{\tau})
\end{aligned}$$

because $\tau > 1$. Hence

$$\lim_{\eta\to 0} \frac{|\mathcal{W}^c_{\gamma,\tau} \cap (1, 1+\eta)|}{|\eta|} = 0\,,$$

and the estimate (2.60) follows. ∎

For ω satisfying only (2.59), the moduli of the eigenvalues of $(L_\omega)_{|W}$ can be bounded from below just as

$$|\omega^2 l^2 - j^2| = |\omega l - j|(\omega l + j) \geq \frac{\gamma}{l^\tau}(\omega l + j) \geq \frac{\gamma \omega}{l^{\tau-1}} \tag{2.62}$$

(actually, for infinitely many (l_n, j_n) the estimate (2.62) is accurate).

As a consequence, the operator

$$(L_\omega)^{-1}_{|W} w = \sum_{j\geq 1, l\geq 0, l\neq j} \frac{w_{l,j}}{-\omega^2 l^2 + j^2} \cos lt \sin jx\,, \quad \forall w \in W\,,$$

does not send $(W, \|\cdot\|)$ into itself.

Actually, by (2.62), the operator $(L_\omega)^{-1}_{|W}$ "loses $(\tau - 1)$ derivatives" and the usual Picard iteration method employed in the standard implicit function theorem cannot lead to an approximation of the solution, since after finitely many steps all derivatives are exhausted.

One might think to work in spaces of C^∞ or analytic functions. For example, in the sequel we shall deal with the space of functions

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\, u_l(x) \mid u_l \in H_0^1((0,\pi),\mathbf{R})\,, u_l(x) = u_{-l}(x) \\ \text{and } \|u\|^2_{\sigma,s} := \sum_{l\in\mathbf{Z}} \exp(2\sigma|l|)(l^{2s}+1)\|u_l\|^2_{H^1} < +\infty \Big\}, \tag{2.63}$$

which are analytic in time ($\sigma > 0$) and valued in $H_0^1((0,\pi),\mathbf{R})$. More precisely, for $\sigma > 0, s \geq 0$, $X_{\sigma,s}$ is the space of all even, 2π-periodic in time functions with values in $H_0^1((0,\pi),\mathbf{R})$ that have a bounded analytic extension in the complex strip $|\mathrm{Im}\, t| < \sigma$ with trace function on $|\mathrm{Im}\, t| = \sigma$ belonging to $H^s(\mathbf{T}, H_0^1((0,\pi),\mathbf{C}))$.

By (2.62), for $w \in W \cap X_{\sigma,s}$, $L_\omega^{-1} w \in X_{\sigma',s}$ just for $\sigma' < \sigma$ and

$$\begin{aligned}
\|L_\omega^{-1} w\|^2_{\sigma',s} &= c \sum_{|l|\neq j} \exp(2\sigma'|l|)(l^{2s}+1)\frac{w^2_{l,j} j^2}{(-\omega^2 l^2 + j^2)^2} \\
&\overset{(2.62)}{\leq} c \sum_{|l|\neq j} \exp(2\sigma'|l|)(l^{2s}+1) l^{2(\tau-1)} \frac{w^2_{l,j} j^2}{\gamma^2} \\
&= c \sum_{|l|\neq j} \exp(-2(\sigma-\sigma')|l|) l^{2(\tau-1)} \exp(2\sigma|l|)(l^{2s}+1)\frac{w^2_{l,j} j^2}{\gamma^2} \\
&\leq C \frac{\|h\|^2_{\sigma,s}}{\gamma^2(\sigma-\sigma')^{2(\tau-1)}}
\end{aligned}$$

because

$$\exp(-2(\sigma-\sigma')y)\, y^{2(\tau-1)} \leq \frac{C(\tau)}{(\sigma-\sigma')^{2(\tau-1)}}\,, \qquad \forall y \geq 0\,.$$

The resulting "Cauchy-type" estimate for the inverse of $(L_\omega)_{|W}$ is

$$\|L_\omega^{-1} w\|_{\sigma',s} \leq \frac{C}{\gamma\,(\sigma - \sigma')^{(\tau-1)}} \|w\|_{\sigma,s}\,, \qquad \sigma' < \sigma\,. \tag{2.64}$$

In this case, however, the estimates for the approximate solutions obtained by the Picard iteration scheme will diverge; see Remark 3.2.

In the next chapter we present another iteration scheme, originally proposed by Nash and Moser, to overcome such difficulties due to the small divisors.

3

A Tutorial in Nash–Moser Theory

3.1 Introduction

The classical implicit function theorem is concerned with the solvability of the equation

$$\mathcal{F}(x, y) = 0\,, \tag{3.1}$$

where $\mathcal{F}\colon X \times Y \to Z$ is a smooth map, X, Y, Z are Banach spaces, and there exists $(x_0, y_0) \in X \times Y$ such that

$$\mathcal{F}(x_0, y_0) = 0\,.$$

If x is close to x_0 we want to solve (3.1) by finding $y = y(x)$.

The main assumption of the classical implicit function theorem is that the partial derivative $(D_y\mathcal{F})(x_0, y_0)\colon Y \to Z$ possesses a *bounded* inverse

$$(D_y\mathcal{F})^{-1}(x_0, y_0) \in \mathcal{L}(Z, Y)\,.$$

Note that if $(D_y\mathcal{F})(x_0, y_0) \in \mathcal{L}(Y, Z)$ is injective and surjective, by the open mapping theorem, the inverse operator $(D_y\mathcal{F})^{-1}(x_0, y_0)\colon Z \to Y$ is automatically continuous.

As we saw in the previous chapter, there are situations in which

$$(D_y\mathcal{F})(x_0, y_0) \text{ has an } \textit{unbounded} \text{ inverse}$$

(for example, the image $(D_y\mathcal{F})(x_0, y_0)\,[Y]$ is only dense in Z).

An approach to this class of problems was proposed by Nash in the pioneering paper [103] for proving that any Riemannian manifold can be isometrically embedded in $\mathbf{R}^N$ for N sufficiently large. Subsequently, Moser [93] highlighted the main features of the technique in an abstract setting, being able to cover problems arising from celestial mechanics and partial differential equations [94]–[96]. Further extensions and applications were made by Gromov [67], by Zehnder [135] to small-divisor problems, by Hörmander [73] to problems in gravitation, by Sergeraert [124] to catastrophe theory, by Schaeffer [120] to free boundary problems in electromagnetics, by Beale [18] in water waves, by Hamilton [69] to foliations, by Klainermann [78]–[79] to Cauchy problems, to mention just a few, showing the power and versatility of the technique.

The main idea is to replace the usual Picard iteration method with a modified Newton iteration scheme. Roughly speaking, the advantage is that since this latter scheme is quadratic (see Remarks 3.2 and 3.7), the iterates converge to the expected solution at a superexponential rate. This accelerated speed of convergence is sufficiently strong to compensate the divergences in the scheme due to the "loss of derivatives."

There are many ways to present the Nash–Moser theorems, according to the applications one has in mind. We shall prove first a very simple "analytic" implicit function theorem (inspired by Theorem 6.1 by Zehnder in [103]; see also [53]) to highlight the main features of the method in an abstract "analytic" setting (i.e., with estimates that can be typically obtained in Banach scales of analytic functions). In our next application to the nonlinear wave equation, we will be able to prove, with a variant of this scheme, the existence of analytic (in time) solutions of (NLW) for positive-measure sets of frequencies.

Next, for completeness, we present also a Nash–Moser theorem in a differentiable setting (i.e., modeled for applications on spaces of functions with finite differentiability such as, for example, Banach scales of Sobolev spaces). To avoid technicalities we present it in the form of an inversion-type theorem as in Moser [93].

3.2 An Analytic Nash–Moser Theorem

Consider three one-parameter families of Banach spaces,

$$X_\sigma\,,\ Y_\sigma\,,\ Z_\sigma\,, \qquad\qquad 0 \le \sigma \le 1\,,$$

with norms $|\cdot|_\sigma$ such that (Banach scales)

$$\forall 0 \le \sigma \le \sigma' \le 1\,, \qquad |x|_\sigma \le |x|_{\sigma'}\,, \quad \forall x \in X_{\sigma'}$$

(analogously for Y_σ, Z_σ), so that

$$\forall 0 \le \sigma \le \sigma' \le 1\,, \qquad X_1 \subseteq X_{\sigma'} \subseteq X_\sigma \subseteq X_0$$

(the same for Y_σ, Z_σ).

EXAMPLE: The Banach spaces of analytic functions

$$X_\sigma := \Big\{ f\colon \mathbf{T}^d \to \mathbf{R}\,,\ f(\varphi) := \sum_k f_k e^{ik\cdot\varphi} \mid |f|_\sigma := \sum_k |f_k| e^{\sigma|k|} < +\infty \Big\}\,.$$

Let

$$\mathcal{F}\colon X_0 \times Y_0 \to Z_0$$

be a mapping defined on the largest spaces of the scales.

Suppose there exists $(x_0, y_0) \in X_1 \times Y_1$ (in the smallest spaces) such that

$$\mathcal{F}(x_0, y_0) = 0 . \tag{3.2}$$

Assume that

$$\mathcal{F}(B_\sigma) \subset Z_\sigma \qquad \forall 0 \leq \sigma \leq 1 , \tag{3.3}$$

where B_σ is the neighborhood of (x_0, y_0)

$$B_\sigma := B_R^\sigma(x_0) \times B_R^\sigma(y_0) \subset X_\sigma \times Y_\sigma$$

and

$$B_R^\sigma(x_0) := \left\{ x \in X_\sigma \mid |x - x_0|_\sigma < R \right\}$$

analogously for $B_R^\sigma(y_0) \subset Y_\sigma$.

We shall make the following hypotheses, in which K and τ are fixed positive constants.

(H1) (Taylor Estimate) $\forall 0 < \sigma \leq 1$, $\forall x \in B_R^\sigma(x_0)$ the map $\mathcal{F}(x, \cdot)\colon B_R^\sigma(y_0) \to Z_\sigma$ is differentiable and $\forall (x, y), (x, y') \in B_\sigma$,

$$\left| \mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y \mathcal{F})(x, y) \left[y' - y \right] \right|_\sigma \leq K |y' - y|_\sigma^2 .$$

Condition (H1) is clearly satisfied if $\mathcal{F}(x, \cdot) \in C^2(B_R^\sigma(y_0), Z_\sigma)$ and $D_{yy}^2 \mathcal{F}(x, \cdot)$ is uniformly bounded for $x \in B_R^\sigma(x_0)$.

(H2) (Right Inverse of loss τ) $\forall 0 < \sigma \leq 1$, $\forall (x, y) \in B_\sigma$ there is a linear operator $L(x, y) \in \mathcal{L}(Z_\sigma, Y_{\sigma'})$, $\forall \sigma' < \sigma$, such that $\forall z \in Z_\sigma$,

$$(D_y \mathcal{F})(x, y) \circ L(x, y) z = z$$

in $Z_{\sigma'}$ and

$$\left| L(x, y) [z] \right|_{\sigma'} \leq \frac{K}{(\sigma - \sigma')^\tau} |z|_\sigma . \tag{3.4}$$

The operator $L(x, y)$ is the right inverse of $(D_y \mathcal{F})(x, y)$ in the sense that

$$(D_y \mathcal{F})(x, y) \circ L(x, y) \text{ is the continuous injection } Z_\sigma \xrightarrow{i} Z_{\sigma'}$$

$\forall \sigma' < \sigma$. Estimate (3.4) is a typical "Cauchy-type" estimate (like (2.64)) for operators acting somewhat like differential operators of order τ in scales of Banach spaces of analytic functions.

Theorem 3.1. *Let $\mathcal{F}$ satisfy (3.2),(3.3), (H1)–(H2). If $x \in B_R^\sigma(x_0)$ for some $\sigma \in (0, 1]$ and $|\mathcal{F}(x, y_0)|_\sigma$ is sufficiently small,*[1] *then there exists a solution*

$$y(x) \in B_R^{\sigma/2}(y_0) \subset Y_{\sigma/2}$$

of the equation

$$\mathcal{F}(x, y(x)) = 0 .$$

[1] Quantified in (3.7); this latter condition defines a neighborhood of x_0 in X_σ.

Proof. We define the Newton iteration scheme

$$\begin{cases} y_{n+1} = y_n - L(x, y_n)\mathcal{F}(x, y_n) , \\ y_0 := y_0 \in Y_1 \subseteq Y_\sigma , \end{cases} \tag{3.5}$$

for $n \geq 0$. Throughout the induction proof we will verify at each step (see the CLAIM below) that y_n belongs to the domain of $\mathcal{F}(x, \cdot)$, $L(x, \cdot)$ and therefore y_{n+1} is well defined.

Since the inverse operator $L(x, y_n)$ "loses analyticity" (hypothesis (H2)), the iterates y_n will belong to larger and larger spaces Y_{σ_n}.

To quantify this phenomenon, let us define the sequence

$$\sigma_0 := \sigma \in (0, 1] \ , \ \sigma_{n+1} := \sigma_n - \delta_n \ ,$$

where the "loss of analyticity" at each step of the iteration is

$$\delta_n := \frac{\delta_0}{n^2 + 1}$$

and $\delta_0 > 0$ is small enough so that the "total loss of analyticity" $\sum_{n \geq 0} \delta_n$ is less than $\sigma/2$, namely

$$\sum_{n \geq 0} \delta_n = \sum_{n \geq 0} \frac{\delta_0}{n^2 + 1} < \frac{\sigma}{2} \tag{3.6}$$

(therefore $\sigma_n > \sigma/2$, $\forall n \geq 0$).

We claim the following:

CLAIM: *Take $\chi := 3/2$ and define*[2]

$$\begin{aligned} \rho &:= \rho(K, R, \tau, \sigma) \\ &:= \min\left\{ \frac{\sqrt{e}}{K}, \min_{n \geq 0} \left(\frac{\delta_{n+1}^\tau}{K^2} \exp((2 - \chi)\chi^n) \right), \frac{R/2}{\sum_{k=0}^\infty \exp(-\chi^k)} \right\} > 0 \, . \end{aligned}$$

If

$$|\mathcal{F}(x, y_0)|_\sigma < \min\left\{ \rho e^{-1} , \frac{\delta_0^\tau}{K} \rho e^{-1}, \frac{\delta_0^\tau}{K} \frac{R}{2} \right\} , \tag{3.7}$$

then the following statements hold for all $n \geq 0$:

(n; 1) $(x, y_n) \in B_{\sigma_n}$ and $|\mathcal{F}(x, y_n)|_{\sigma_n} \leq \rho \exp(-\chi^n)$,
(n; 2) $|y_{n+1} - y_n|_{\sigma_{n+1}} \leq \rho \exp(-\chi^n)$,
(n; 3) $|y_{n+1} - y_0|_{\sigma_{n+1}} < R/2$.

[2] We have $\rho > 0$ because the sequence of positive numbers $\delta_{n+1}^\tau \exp((2 - \chi)\chi^n) = \delta_0^\tau (\exp(\frac{1}{2}(3/2)^n))/((1 + (n + 1)^2)^\tau)$ goes to $+\infty$ as $n \to +\infty$.

Before proving the claim, let us conclude the proof of Theorem 3.1.

By statement $(n; 2)$ the sequence $y_n \in Y_{\sigma/2}$ is a Cauchy sequence (in the largest space $Y_{\sigma/2}$). Indeed, for any $n > m$,

$$\begin{aligned} |y_n - y_m|_{\sigma/2} &\leq \sum_{k=m}^{n-1} |y_{k+1} - y_k|_{\sigma/2} \leq \sum_{k=m}^{n-1} |y_{k+1} - y_k|_{\sigma_{k+1}} \\ &\stackrel{(k;2)}{\leq} \sum_{k=m}^{n-1} \rho \exp(-\chi^k) \to 0 \ \text{ for } \ n, m \to +\infty\,. \end{aligned}$$

Hence y_n converges in $Y_{\sigma/2}$ to some $y(x) \in Y_{\sigma/2}$. Actually, $y(x) \in \overline{B^{\sigma/2}_{R/2}(y_0)} \subset B^{\sigma/2}_R$ by $(n; 3)$. Finally, by the continuity of $\mathcal{F}$ with respect to the second variable and $(n; 1)$,

$$\mathcal{F}(x, y(x)) = \lim_{n\to\infty} \mathcal{F}(x, y_n) = 0\,,$$

implying that $y(x)$ is a solution of $\mathcal{F}(x, y) = 0$.

Let us now prove the claim. Its proof proceeds by induction. First, let us verify it for $n = 0$. It reduces to the smallness condition (3.7) for $|\mathcal{F}(x, y_0)|_\sigma$.

(0; 1) By assumption $x \in B^\sigma_R(x_0)$, so that $(x, y_0) \in B_{\sigma_0} := B_\sigma$. By (3.3) we have that $\mathcal{F}(x, y_0) \in Z_\sigma$, and $|\mathcal{F}(x, y_0)|_\sigma \leq \rho \mathrm{e}^{-1}$ follows by (3.7).

(0; 2)–(0; 3) Since $(x, y_0) \in B_\sigma$, by (3.5) and (H2),

$$|y_1 - y_0|_{\sigma_1} = |L(x, y_0)\mathcal{F}(x, y_0)|_{\sigma_1} \leq \frac{K}{(\sigma_0 - \sigma_1)^\tau} |\mathcal{F}(x, y_0)|_\sigma\,.$$

Under the smallness condition (3.7) we have verified both (0; 2) and (0; 3).

Now suppose $(n; 1)$–$(n; 3)$ are true. By $(n; 3)$,

$$y_{n+1} \in B^{\sigma_{n+1}}_R(y_0)\,,$$

and so

$$(x, y_{n+1}) \in B_{\sigma_{n+1}}\,.$$

Hence $\mathcal{F}(x, y_{n+1}) \in Z_{\sigma_{n+1}}$ (by (3.3)) and, by (H2),

$$y_{n+2} := y_{n+1} - L(x, y_{n+1})\mathcal{F}(x, y_{n+1}) \in Y_{\sigma_{n+2}}$$

is well defined.

Set for brevity

$$Q(y, y') := \mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y\mathcal{F})(x, y)\left[y' - y\right]\,. \tag{3.8}$$

By a Taylor expansion,

$$
\begin{aligned}
|\mathcal{F}(x, y_{n+1})|_{\sigma_{n+1}} &= \Big|\mathcal{F}(x, y_n) + (D_y\mathcal{F})(x, y_n)\,[y_{n+1} - y_n] + Q(y_n, y_{n+1})\Big|_{\sigma_{n+1}} \\
&\overset{(3.5)}{=} |Q(y_n, y_{n+1})|_{\sigma_{n+1}} \overset{(\mathrm{H1})}{\le} K|y_{n+1} - y_n|^2_{\sigma_{n+1}} \qquad (3.9) \\
&\overset{(n;2)}{\le} K\rho^2 \exp(-2\chi^n)\,. \qquad (3.10)
\end{aligned}
$$

By (3.10) the claim $(n + 1; 1)$ is verified whenever

$$K\rho^2 \exp(-2\chi^n) < \rho \exp(-\chi^{n+1})\,,$$

which holds true for any $n \ge 0$ if

$$\rho < \min_{n\ge 0}\left(\frac{1}{K}\exp((2-\chi)\chi^n)\right) = \frac{\sqrt{\mathrm{e}}}{K}\,. \qquad (3.11)$$

Now

$$
\begin{aligned}
|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} &\overset{(3.5)}{=} |L(x, y_{n+1})\mathcal{F}(x, y_{n+1})|_{\sigma_{n+2}} \\
&\overset{(\mathrm{H2})}{\le} \frac{K}{(\sigma_{n+1} - \sigma_{n+2})^\tau}|\mathcal{F}(x, y_{n+1})|_{\sigma_{n+1}} \\
&\overset{(3.9)}{\le} \frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^\tau}|y_{n+1} - y_n|^2_{\sigma_{n+1}} \qquad (3.12) \\
&\overset{(n;2)}{\le} \frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^\tau}\rho^2 \exp(-2\chi^n)
\end{aligned}
$$

and therefore the claim $(n + 1; 2)$ is verified whenever

$$\frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^\tau}\rho^2 \exp(-2\chi^n) < \rho \exp(-\chi^{n+1})\,,$$

which holds for any $n \ge 0$, if

$$\rho < \min_{n\ge 0}\left(\frac{\delta^\tau_{n+1}}{K^2}\exp((2-\chi)\chi^n)\right). \qquad (3.13)$$

Finally,

$$
\begin{aligned}
|y_{n+2} - y_0|_{\sigma_{n+2}} &\le \sum_{k=0}^{n+1}|y_{k+1} - y_k|_{\sigma_{n+2}} \le \sum_{k=0}^{n+1}|y_{k+1} - y_k|_{\sigma_{k+1}} \\
&\overset{(k;2)}{\le} \sum_{k=0}^{n+1}\rho\exp(-\chi^k) < \rho\sum_{k=0}^{\infty}\exp(-\chi^k)\,,
\end{aligned}
$$

which implies $(n + 1; 3)$ assuming

$$\rho < \frac{R/2}{\sum_{k=0}^{\infty}\exp(-\chi^k)}\,. \qquad (3.14)$$

In conclusion, if $\rho > 0$ is small enough (depending on K, τ, R, σ), according to (3.11)–(3.14) the claim is proved.

This completes the proof. ■

Remark 3.2. The key point of the Nash–Moser scheme is the estimate

$$|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} \le \frac{K^2}{\delta_{n+1}^{\tau}} |y_{n+1} - y_n|^2_{\sigma_{n+1}} ; \tag{3.15}$$

see (3.12). Even though $\delta_n \to 0$, this quadratic estimate ensures that the sequence of numbers $|y_{n+1} - y_n|_{\sigma_{n+1}}$ tends to zero at a superexponential rate (see $(n; 2)$) if $|y_1 - y_0|_{\sigma_1}$ is sufficiently small. Note that the Picard iteration scheme would yield just $|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} \le C\delta_{n+1}^{-\tau} |y_{n+1} - y_n|_{\sigma_{n+1}}$, i.e., the divergence of the estimates.

Clearly, the drawback to get (3.15) is to invert the linearized operators in a *whole* neighborhood of (x_0, y_0); see (H2). This is the most difficult step to apply the Nash–Moser method in concrete situations; see, e.g., Section 4.5.

Remark 3.3. The hypotheses in Theorem 3.1 can be considerably weakened; see [103]. For example, in (H1) one could assume a loss of analyticity[3] also in the quadratic part of the Taylor expansion

$$\Big|\mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y\mathcal{F})(x, y)\,[y' - y]\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\alpha}} |y' - y|^2_{\sigma}$$

$\forall \sigma' < \sigma$ and some $\alpha > 0$ (independent of σ).

Furthermore, one could assume in (H2) just[4] the existence of an "approximate right inverse," namely $\forall z \in Z_\sigma$,

$$\Big|\Big((D_y\mathcal{F})(x, y) \circ L(x, y) - I\Big)\,[z]\Big)\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\tau}} |\mathcal{F}(x, y)|_\sigma |z|_\sigma . \tag{3.16}$$

Note that $L(x, y)$ is an exact inverse at the solutions $\mathcal{F}(x, y) = 0$.

Furthermore, in the statement of Theorem 3.1 it is possible to get better and quantitative estimates.

Since we have not assumed the existence of the left inverse of $(D_y\mathcal{F})(x, y)$ in the assumptions of Theorem 3.1, uniqueness of the solution $y(x)$ cannot be expected (it could fail also in the linear problem).

Local uniqueness follows assuming the existence of a left inverse:

(H2)′ $\forall 0 < \sigma \le 1$, $\forall (x, y) \in B_\sigma$ there is a linear operator $\xi(x, y) \in \mathcal{L}(Z_\sigma, Y_{\sigma'})$, $\forall \sigma' < \sigma$ such that $\forall h \in Y_\sigma$,

$$\xi(x, y) \circ (D_y\mathcal{F})(x, y)\,[h] = h$$

in $Y_{\sigma'}$, and $\forall z \in Z_\sigma$,

$$\Big|\xi(x, y)\,[z]\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\tau}} |z|_\sigma . \tag{3.17}$$

[3] In the application considered in the next chapter the quadratic part Q satisfies (H1); i.e., it does not lose regularity.

[4] In the application to the equation (NLW) we can find an inverse operator satisfying (H2).

The operator $\xi(x,y)$ is the left inverse of $(D_y\mathcal{F})(x,y)$ in the sense that $\xi(x,y)\circ(D_y\mathcal{F})(x,y)$ is the continuous injection $Y_\sigma \stackrel{i}{\to} Y_{\sigma'}, \forall\sigma' < \sigma$.

Theorem 3.4. (Uniqueness) *Let $\mathcal{F}$ satisfy (3.3),* **(H1)–(H2)$'$**. *Let (x,y), $(x,y') \in B_\sigma$ be solutions of $\mathcal{F}(x,y) = 0$, $\mathcal{F}(x,y') = 0$. If $|y-y'|_\sigma$ is small enough (depending on K, τ, σ) then $y = y'$ in $Y_{\sigma/2}$.*

Proof. Setting $h := y - y' \in Y_\sigma$, we have

$$\begin{aligned} |h|_{\sigma'} &\stackrel{(\mathrm{H2})'}{=} |\xi(x,y)\circ(D_y\mathcal{F})(x,y)\,[h]\,|_{\sigma'} \\ &\stackrel{(3.17)}{\le} \frac{K}{(\sigma-\sigma')^\tau}|(D_y\mathcal{F})(x,y)\,[h]\,|_\sigma \\ &\le \frac{K}{(\sigma-\sigma')^\tau}|Q(y,y')|_\sigma\,, \end{aligned} \tag{3.18}$$

since $\mathcal{F}(x,y) = 0$, $\mathcal{F}(x,y') = 0$, and recalling the definition of $Q(y,y')$ in (3.8).

By (3.18) and **(H1)** we get

$$|h|_{\sigma'} \stackrel{(\mathrm{H1})}{\le} \frac{K^2}{(\sigma-\sigma')^\tau}|y'-y|_\sigma^2 = \frac{K^2}{(\sigma-\sigma')^\tau}|h|_\sigma^2\,, \quad \forall\sigma' < \sigma\,,$$

whence for $\sigma' := \sigma_{n+1}$, $\sigma := \sigma_n$, $\delta_n := \sigma_n - \sigma_{n+1}$,

$$|h|_{\sigma_{n+1}} \le K^2\delta_n^{-\tau}|h|_{\sigma_n}^2\,, \quad \forall n \ge 0\,.$$

These last estimates imply that if $|h|_\sigma = |y-y'|_\sigma$ ($\sigma = \sigma_0$) is sufficiently small (depending on K, τ, σ), then $|h|_{\sigma_n} \le \rho e^{-\chi^n}$, $\forall n \ge 0$, and therefore $|h|_{\sigma/2} = 0$, namely $h = y - y' = 0$ in $Y_{\sigma/2}$. ∎

3.3 A Differentiable Nash–Moser Theorem

The iterative scheme (3.5) cannot work to prove a Nash–Moser implicit function theorem in spaces, say, of class C^k, because due to the loss of derivatives of the inverse linearized operators, after a fixed number of iterations, all derivatives will be exhausted. The scheme has to be modified by applying a sequence of "smoothing" operators that regularize $y_{n+1} - y_n$ at each step.

To avoid technicalities, we present the ideas of the Nash–Moser differentiable theory in the form of an inversion-type theorem (as in Moser [93]) rather than an implicit-function-type theorem.

To make it precise, consider a Banach scale $(Y_s)_{s\ge0}$ satisfying

$$Y_{s'} \subset Y_s \subset Y_0\,, \quad \forall s' \ge s \ge 0$$

(with norms $|y|_s \le |y|_{s'}$, $\forall y \in Y_{s'}$), equipped with a family of "smoothing" linear operators

$$S(t)\colon Y_0 \to Y_\infty := \bigcap_{s\ge 0} Y_s\,, \qquad t \ge 0\,,$$

such that

$$|S(t)u|_{s+r} \le C_{s,r}\, t^r\, |u|_s\,, \qquad \forall u \in Y_s\,, \tag{3.19}$$

$$|(I - S(t))u|_s \le C_{s,r}\, t^{-r}\, |u|_{s+r}\,, \qquad \forall u \in Y_{s+r}\,, \tag{3.20}$$

for some positive constants $C_{s,r}$. The meaning of (3.20) is that functions u that are already smooth ($u \in Y_{s+r}$), are approximated by $S(t)u$, in the weaker norm $|\ |_s$, with a higher degree of accuracy.

For the construction of these smoothing operators for concrete Banach scales, see for example Schwartz [122] or Zehnder in [103] (via convolutions with suitable "mollifiers").

Remark 3.5. Estimates (3.19)–(3.20) are the usual ones in the Sobolev scale

$$Y_s := \Big\{ f(\varphi) := \sum_k f_k e^{ik\cdot\varphi} \ \Big|\ |f|_s^2 := \sum_k |f_k|^2 (1 + |k|^{2s}) < +\infty \Big\}$$

for the projector S_N on the first N Fourier modes

$$S_N \Big(\sum_k f_k e^{ik\cdot x} \Big) := \sum_{|k|\le N} f_k e^{ik\cdot x}$$

(when $t := N$ is an integer).

3.1. Exercise: On a Banach scale $(X_s)_{s\ge 0}$ equipped with smoothing operators $(S(t))_{t\ge 0}$, the following convexity inequality holds: for all $0 \le \lambda_1 \le \lambda_2$, $\alpha \in [0, 1]$, and $u \in X_{\lambda_2}$,

$$|u|_\lambda \le K_{\lambda_1,\lambda_2} |u|_{\lambda_1}^{1-\alpha}\, |u|_{\lambda_2}^{\alpha}\,, \quad \lambda = (1-\alpha)\lambda_1 + \alpha\lambda_2\,, \tag{3.21}$$

for some positive constant K_{λ_1,λ_2}.

We make the following assumptions, where α, K, τ are fixed positive constants.

(H1) (Tame estimate) $\mathcal{F}\colon Y_{s+\alpha} \to Y_s$, $\forall s \ge 0$, satisfies

$$|\mathcal{F}(y)|_s \le K(1 + |y|_{s+\alpha})\,, \quad \forall y \in Y_{s+\alpha}\,.$$

Differential operators $\mathcal{F}$ of order α satisfy this "tame" property, i.e., $|\mathcal{F}(y)|_s$ grows at most linearly with the higher norm $|\cdot|_{s+\alpha}$; see [103], [69]. This apparently surprising fact follows by the interpolation inequalities (3.21) (as the Gagliardo [64]–Nirenberg [104]–Moser [95] interpolation estimates in Sobolev spaces; see [127] for a modern account).

(H2) (Taylor estimate) $\mathcal{F}\colon Y_{s+\alpha} \to Y_s$, $\forall s \ge 0$, is differentiable and

$$\begin{cases} |(D\mathcal{F})(y)\,[h]\,|_s \le K|h|_{s+\alpha}\,, \\ \Big|\mathcal{F}(y') - \mathcal{F}(y) - (D\mathcal{F})(y)\big[y' - y\big]\Big|_s \le K|y' - y|_{s+\alpha}^2\,. \end{cases}$$

(H3) (Right Inverse of loss τ) $\forall y \in Y_\infty$ there is a linear operator $L(y) \in \mathcal{L}(Y_{s+\tau}, Y_s)$, $\forall s \geq 0$, i.e.,

$$|L(y)\,[h]\,|_s \leq K|h|_{s+\tau}\,, \quad \forall h \in Y_{s+\tau}\,,$$

such that

$$D\mathcal{F}(y) \circ L(y)\,[h] = h\,, \qquad \forall h \in Y_{s+\tau}\,,$$

in $Y_{s-\alpha}$ (the operator $D\mathcal{F}(y) \circ L(y)$ is the inclusion $i\colon Y_{s+\tau} \to Y_{s-\alpha}$).

Hypotheses **(H1)–(H3)** state, roughly, that $\mathcal{F}$, $D\mathcal{F}$, respectively L, act somewhat as differential operators of order α, respectively τ.

Theorem 3.6. *Let $\mathcal{F}$ satisfy (H1)–(H3) and fix any $s_0 > \alpha + \tau$. If $|\mathcal{F}(0)|_{s_0+\tau}$ is sufficiently small (depending on α, τ, K, s_0) then there exists a solution $y \in Y_{s_0}$ of the equation $\mathcal{F}(y) = 0$.*

Proof. Consider the iterative scheme

$$\begin{cases} y_{n+1} = y_n - S(N_n)L(y_n)\mathcal{F}(y_n)\,, \\ y_0 := 0\,, \end{cases} \tag{3.22}$$

where

$$N_n := \exp(\lambda\chi^n)\,, \quad N_{n+1} = N_n^{\chi}\,, \quad \chi := 3/2\,,$$

for some λ large enough, depending on α, τ, K, s_0, to be chosen later.

By (3.22), the increment $y_{n+1} - y_n$ is in Y_∞, $\forall n \geq 0$, and therefore, $y_n \in Y_\infty$, $\forall n \geq 0$ (because $y_0 := 0 \in Y_\infty$). Furthermore,

$$\begin{aligned} |y_{n+1} - y_n|_{s_0} &\overset{(3.22)}{=} |S(N_n)L(y_n)\mathcal{F}(y_n)|_{s_0} \overset{(3.19)}{\leq} C_0 N_n^{\alpha+\tau}|L(y_n)\mathcal{F}(y_n)|_{s_0-\alpha-\tau} \\ &\overset{(H3)}{\leq} C_0 N_n^{\alpha+\tau} K|\mathcal{F}(y_n)|_{s_0-\alpha}\,, \end{aligned} \tag{3.23}$$

where $C_0 := C_{s_0-\alpha-\tau,\alpha+\tau}$ is the constant from (3.19).

By a Taylor expansion, for $n \geq 1$, setting for brevity $Q(y; y') := \mathcal{F}(y') - \mathcal{F}(y) - D\mathcal{F}(y)\left[y' - y\right]$,

$$\begin{aligned} |\mathcal{F}(y_n)|_{s_0-\alpha} &\leq |\mathcal{F}(y_{n-1}) + D\mathcal{F}(y_{n-1})\left[y_n - y_{n-1}\right]|_{s_0-\alpha} + |Q(y_{n-1}, y_n)|_{s_0-\alpha} \\ &\overset{(3.22)}{=} |D\mathcal{F}(y_{n-1})(I - S(N_{n-1}))L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0-\alpha} \\ &\quad + |Q(y_{n-1}, y_n)|_{s_0-\alpha} \\ &\overset{(H2)}{\leq} K|(I - S(N_{n-1}))L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0} + K|y_n - y_{n-1}|_{s_0}^2 \\ &\overset{(3.20)}{\leq} KC_{s_0,\beta}N_{n-1}^{-\beta}B_{n-1} + K|y_n - y_{n-1}|_{s_0}^2\,, \end{aligned} \tag{3.24}$$

where $B_{n-1} := |L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0+\beta}$.

By (3.23) and (3.24) we deduce

$$|y_{n+1} - y_n|_{s_0} \le C_1 N_n^{\alpha+\tau} N_{n-1}^{-\beta} B_{n-1} + C_1 N_n^{\alpha+\tau} |y_n - y_{n-1}|_{s_0}^2 \tag{3.25}$$

for some positive $C_1 := C(\alpha, \tau, s_0, K)$.

To prove, by (3.25), the superexponential smallness of $|y_{n+1} - y_n|_{s_0}$, the main issue is to give an a priori estimate for the divergence of the B_n independent of β.

For $n \ge 0$ we have

$$B_n := |L(y_n)\mathcal{F}(y_n)|_{s_0+\beta} \overset{\text{(H3)}}{\le} K|\mathcal{F}(y_n)|_{s_0+\beta+\tau} , \tag{3.26}$$

and for $n \ge 1$, writing $y_n = \sum_{k=1}^n (y_k - y_{k-1})$,

$$\begin{aligned} B_n &\overset{\text{(H1)}}{\le} K^2(1 + |y_n|_{s_0+\beta+\tau+\alpha}) \le K^2\Big(1 + \sum_{k=1}^n |y_k - y_{k-1}|_{s_0+\beta+\tau+\alpha}\Big) \\ &\overset{(3.22)}{=} K^2\Big(1 + \sum_{k=1}^n |S(N_{k-1})L(y_{k-1})\mathcal{F}(y_{k-1})|_{s_0+\beta+\tau+\alpha}\Big) \\ &\overset{(3.19)}{\le} K^2\Big(1 + \sum_{k=1}^n C_2 N_{k-1}^{\tau+\alpha} |L(y_{k-1})\mathcal{F}(y_{k-1})|_{s_0+\beta}\Big) \\ &\le C_3\Big(1 + \sum_{k=0}^{n-1} N_k^{\tau+\alpha} B_k\Big) , \end{aligned} \tag{3.27}$$

where $C_2 := C_{s_0+\beta,r+\alpha}$ is the constant from (3.19) and $C_3 := K^2 \max\{1, C_2\}$.

We claim the following:

CLAIM: *Take* $\beta := 15(\alpha + \tau)$ *and suppose*

$$|\mathcal{F}(0)|_{s_0+\tau} < e^{-\lambda 4(\alpha+\tau)}/K C_{s_0,0} . \tag{3.28}$$

There is $\lambda := \lambda(\tau, \alpha, K, s_0) \ge 1$ *such that the following statements hold for all* $n \ge 0$*:*

(n; 1) $B_n \le N_n^\nu = \exp(\lambda\chi^n\nu)$, $\quad \nu := 4(\tau + \alpha)$,
(n; 2) $|y_{n+1} - y_n|_{s_0} \le N_n^{-\nu} = \exp(-\lambda\chi^n\nu)$.

Statement (0; 1) is verified by

$$B_0 := |L(0)\mathcal{F}(0)|_{s_0+\beta} \overset{\text{(H3)}}{\le} K|\mathcal{F}(0)|_{s_0+\beta+\tau} \le e^{\lambda\nu} ,$$

which holds for $\lambda := \lambda(s_0, \alpha, \tau, K)$ large enough.

Statement (0; 2) follows by

$$\begin{aligned} |y_1 - y_0|_{s_0} &\overset{(3.22)}{=} |S(N_0)L(0)\mathcal{F}(0)|_{s_0} \overset{(3.19)}{\le} C_{s_0,0}|L(0)\mathcal{F}(0)|_{s_0} \\ &\overset{\text{(H3)}}{\le} C_{s_0,0}K|\mathcal{F}(0)|_{s_0+\tau} \overset{(3.28)}{<} e^{-\lambda\nu} . \end{aligned}$$

Now suppose (n; 1)–(n; 2) are true. To prove ($n + 1$; 1) write

$$
\begin{aligned}
B_{n+1} &\overset{(3.27)}{\leq} C_3\Big(1+\sum_{k=0}^{n} N_k^{\tau+\alpha} B_k\Big) \overset{(n;1)}{\leq} C_3\Big(1+\sum_{k=0}^{n}\exp((\tau+\alpha+\nu)\lambda\chi^k)\Big) \\
&= C_3\Big(1+\exp((\tau+\alpha+\nu)\lambda\chi^n)\sum_{k=0}^{n}\exp(-(\tau+\alpha+\nu)\lambda(\chi^n-\chi^k))\Big) \\
&\leq C_3\Big(1+\exp((\tau+\alpha+\nu)\lambda\chi^n)\sum_{k=0}^{n}\exp(-5(\tau+\alpha)(\chi^n-\chi^k))\Big) \\
&\leq C_4\exp((\tau+\alpha+\nu)\lambda\chi^n) < \exp(\nu\lambda\chi^{n+1})
\end{aligned}
$$

for some $C_4 := C_4(\alpha, \tau, K, s_0) > 0$ and $\lambda := \lambda(\alpha, \tau, K, s_0) \geq 1$ sufficiently large (because $\nu(\chi - 1) > \tau + \alpha$).

Remark 3.7. The main novelty with respect to the analytic scheme—compare (3.25) with (3.15)—is to prove that the term $N_n^{\alpha+\tau} N_{n-1}^{-\beta} B_{n-1}$ in (3.25) is superexponentially small. This follows, for β large, by $(n; 1)$, implying that $|y_{n+1} - y_n|_{s_0}$ still converges to zero at a superexponential rate if $|y_1 - y_0|_{s_0}$ is sufficiently small; see statement $(n; 2)$.

Let us prove $(n + 1; 2)$. Recalling that $N_n := \exp(\lambda\chi^n)$, we have

$$
\begin{aligned}
|y_{n+2} - y_{n+1}|_{s_0} &\overset{(3.25)}{\leq} C_1\exp(\lambda(\alpha+\tau)\chi^{n+1})\exp(-\lambda\beta\chi^n)B_n \\
&\quad + C_1\exp(\lambda(\alpha+\tau)\chi^{n+1})|y_{n+1}-y_n|_{s_0}^2 \\
&\overset{(n;1),\,(n;2)}{\leq} C_1\exp(\lambda(\alpha+\tau)\chi^{n+1})\exp(-\lambda\beta\chi^n)\exp(\lambda\nu\chi^n) \\
&\quad + C_1\exp(\lambda(\alpha+\tau)\chi^{n+1})\exp(-2\nu\lambda\chi^n) \\
&\leq \exp(-\lambda\chi^{n+1}\nu)
\end{aligned}
$$

once we impose

$$
C_1\exp(\lambda\chi^n(\chi(\alpha+\tau)-\beta+\nu)) < \frac{\exp(-\lambda\chi^{n+1}\nu)}{2}
$$

and

$$
C_1\exp(\lambda\chi^n(\chi(\alpha+\tau)-2\nu)) < \frac{\exp(-\lambda\chi^{n+1}\nu)}{2}.
$$

These inequalities are satisfied, for λ large enough depending on α, τ, K, s_0, because

$$
\beta - \nu(1+\chi^2) - \chi(\alpha+\tau) > 0 \quad \text{and} \quad (2-\chi)\nu - \chi(\alpha+\tau) > 0
$$

for $\beta := 15(\alpha + \tau)$, $\nu := 4(\alpha + \tau)$, $\chi = 3/2$.

This concludes the proof of the claim.

By $(n; 2)$ the sequence y_n is a Cauchy sequence in Y_{s_0} and therefore $y_n \to y \in Y_{s_0}$. By (3.24), $(n; 1)$–$(n; 2)$, $|\mathcal{F}(y_n)|_{s_0-\alpha} \to 0$, and therefore, by the continuity of $\mathcal{F}(\cdot)$, we deduce $\mathcal{F}(y) = 0$. ∎

Remark 3.8. Clearly much weaker conditions could be assumed. First of all, conditions (H1)–(H3) need to hold just on a neighborhood of $y_0 = 0$. Next, we could allow the constant $K := K(|\ |_{s_0})$ to depend on the weaker norm $|\cdot|_{s_0}$. The inverse could be substituted by an approximate right inverse as in (3.16).

4

Application to the Nonlinear Wave Equation

We want now to prove existence of periodic solutions of the completely resonant nonlinear wave equation (NLW)

$$\begin{cases} u_{tt} - u_{xx} = a(x)u^p \,, \\ u(t,0) = u(t,\pi) = 0 \,, \end{cases} \tag{4.1}$$

where $p \geq 2$, $p \in \mathbf{N}$, for sets of frequencies ω close to 1 with asymptotically full measure.

Remark 4.1. Under spatial periodic boundary conditions Bourgain [37] proved, when $f = u^3 + O(u^4)$, the existence of periodic solutions, branching off exact traveling wave solutions of $u_{tt} - u_{xx} + u^3 = 0$. Recently, Yuan [132] has proved, still for periodic boundary conditions, the existence of certain types of quasiperiodic solutions. For Dirichlet boundary conditions, periodic solutions have been proved by Gentile–Mastropietro–Procesi [65] for analytic $f = u^3 + O(u^5)$, using the Lindsted series techniques.

Normalizing the period $t \to \omega t$ and rescaling the amplitude $u \to \delta u$, $\delta > 0$, as in Chapter 2, we look for 2π-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} = \varepsilon a(x)u^p \,, \\ u(t,0) = u(t,\pi) = 0 \,, \end{cases} \tag{4.2}$$

where $\varepsilon := \delta^{p-1}$.

In order to apply a Nash–Moser scheme, we look for solutions of (4.2) in the space $X_{\sigma,s}$ introduced in (2.63), formed by functions analytic in time ($\sigma > 0$) and valued in $H_0^1(0,\pi)$. In [28] existence of periodic solutions with $\sigma = 0$, i.e., just Sobolev in time, has been obtained for just differentiable nonlinearities, with a scheme similar to that of Theorem 3.6.

We shall fix $s > 1/2$ so that $X_{\sigma,s}$ is a multiplicative Banach algebra (see Appendix E), namely

$$\|u_1 u_2\|_{\sigma,s} \le C \|u_1\|_{\sigma,s} \|u_2\|_{\sigma,s} , \qquad \forall u_1, u_2 \in X_{\sigma,s} ,$$

and the Nemitsky operator induced on $X_{\sigma,s}$ by $f(x,u) = a(x)u^p$ satisfies, taking $a(x) \in H^1(0,\pi)$,

$$\|f(u)\|_{\sigma,s} \le C \|u\|^p_{\sigma,s} , \tag{4.3}$$

and it is in $C^\infty(X_{\sigma,s}, X_{\sigma,s})$ (actually it is analytic; see [111]).

Let us explain the choice of this space.

Remark 4.2. For nonodd nonlinearities, it is not possible in general to find nontrivial solutions of (4.1) valued in

$$u(t,\cdot) \in Y := \Big\{ u(x) = \sum_{j\ge 1} u_j \sin(jx) \mid \sum_j \exp(2aj) j^{2\rho} |u_j|^2 < +\infty \Big\}$$

where $a > 0$, $\rho \ge 0$. Consider, for example,

$$\begin{cases} u_{tt} - u_{xx} = u^2 , \\ u(t,0) = u(t,\pi) = 0 . \end{cases} \tag{4.4}$$

Since $u(t,0) = 0$, $\forall t$, then $u_{tt}(t,0) = 0$, and by (4.4), $u_{xx}(t,0) = 0$, $\forall t$. Differentiating twice (4.4) with respect to x and using that $u(t,0) = 0$, $u_{ttxx}(t,0) = 0$, we deduce

$$u_{xxxx}(t,0) + 2u_x^2(t,0) = 0 , \ \forall t .$$

Now $u_{xxxx}(t,0) = 0$, because all the even derivatives of any function in Y vanish at $x = 0$. Hence $u_x^2(t,0)$ =0, $\forall t$. Differentiating again the equation, we can prove that $\partial_x^k u(t,0) = 0$, $\forall k$, $\forall t$. Hence, by the analyticity of $u(t,\cdot) \in Y$, $u \equiv 0$.

Projecting (4.2) according to the orthogonal decomposition

$$X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s}) ,$$

where V is defined in (2.14) and

$$W := \Big\{ w = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\, w_l(x) \in X_{0,s} \mid w_{-l} = w_l \ \text{and} \int_0^\pi w_l(x)\sin(lx)\,\mathrm{d}x = 0, \ \forall l \in \mathbf{Z} \Big\} ,$$

yields (using that $v_{tt} = v_{xx}$)

$$\begin{cases} \dfrac{(\omega^2 - 1)}{2} \Delta v = \varepsilon \Pi_V f(v + w) & \textit{bifurcation equation} , \\ L_\omega w = \varepsilon \Pi_W f(v + w) & \textit{range equation} , \end{cases} \tag{4.5}$$

where

$$\Delta v := v_{xx} + v_{tt} , \quad L_\omega := \omega^2 \partial_{tt} - \partial_{xx} . \tag{4.6}$$

As explained at the end of Chapter 2, in order to find periodic solutions for *positive*-measure sets of frequencies ω, we have to solve the range equation

$$L_\omega w - \varepsilon \Pi_W f(v + w) = 0\,,$$

finding $w = w(\varepsilon, v)$ via a Nash–Moser-type iteration scheme.

There are two main difficulties in this program:

(i) Prove the invertibility of the linearized operators

$$h \mapsto \mathcal{L}(\varepsilon, v, w)\,[h] := L_\omega h - \varepsilon \Pi_W f'(v + w)h$$

obtained at any step of the Nash–Moser iteration (with a loss of analyticity as in hypothesis (H2) of Theorem 3.1).

This invertibility property does *not* hold for all the values of the parameters (ε, v). The eigenvalues $\{\lambda_{lj}(\varepsilon, v),\ l \geq 0,\ j \geq 1\}$ of $\mathcal{L}(\varepsilon, v, w)$ are, in general, dense in $\mathbf{R}$ (as the spectrum of the unperturbed operator L_ω), and they depend very sensitively on (ε, v). The linear operator $\mathcal{L}(\varepsilon, v, w)$ will be invertible only outside the "resonant web" of parameters (ε, v) where at least one $\lambda_{lj}(\varepsilon, v)$ is zero. As a consequence, the Nash–Moser iteration scheme will converge to a solution $w(\varepsilon, v)$ just on a complicated Cantor-like set; see Theorem 4.8.

The presence of these "Cantor gaps" will have painful implications for solving next the bifurcation equation; see Remark 4.28.

The invertibility of the linear operator $\mathcal{L}(\varepsilon, v, w)$ is obtained by the new method developed in [26], which is different from the approach of Craig–Wayne [51] and Bourgain [35]–[36]. The method in [26] requires less regularity, and it does not demand oddness assumptions on the nonlinearity.

The second main difficulty that appears is related to the fact that we are dealing with a completely resonant PDE, and the solutions $v \in V$ of the linear equation (2.8) form an *infinite*-dimensional space:

(ii) If $v \in V \cap X_{\sigma,s}$, then the solution $w(\varepsilon, v)$ of the range equation obtained through the Nash–Moser iteration will have a lower regularity, e.g., $w(\varepsilon, v) \in X_{\sigma/2,s}$. Therefore in solving next the bifurcation equation for $v \in V$, the best estimate we can obtain is $v \in V \cap X_{\sigma/2,s+2}$, which makes the scheme incoherent.

This second problem does not arise for nonresonant or partially resonant Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = a_2(x)u^2 + \cdots$, where the bifurcation equation is finite-dimensional [51]–[52].

4.1 The Zeroth-Order Bifurcation Equation

To deal with this infinite-dimensional bifurcation problem we shall consider here the simplest situation, which occurs when

$$\Pi_V(a(x)v^p) \not\equiv 0\,, \tag{4.7}$$

or equivalently,

$$\exists v \in V \quad \text{such that} \quad \int_\Omega a(x)v^{p+1} \neq 0\,. \tag{4.8}$$

For definiteness we shall assume that $\exists v \in V$ such that $\int_\Omega a(x)v^{p+1} > 0$. In this case we set the "frequency–amplitude" relation[1]

$$\frac{\omega^2 - 1}{2} = -\varepsilon\,,$$

and system (4.5) becomes

$$\begin{cases} -\Delta v = \Pi_V f(v + w) & \textit{bifurcation equation}\,, \\ L_\omega w = \varepsilon \Pi_W f(v + w) & \textit{range equation}\,. \end{cases} \tag{4.9}$$

When $\varepsilon = 0$, system (4.9) reduces to $w = 0$ and to the "0th-order bifurcation equation"

$$-\Delta v = \Pi_V(a(x)v^p)\,, \tag{4.10}$$

which is the Euler–Lagrange equation of the functional $\Phi_0 \colon V \mapsto \mathbf{R}$,

$$\Phi_0(v) := \frac{\|v\|^2_{H^1}}{2} - \int_\Omega a(x)\frac{v^{p+1}}{p+1}\,, \tag{4.11}$$

where $\|v\|^2_{H^1} := \int_\Omega v_t^2 + v_x^2$.

Remark 4.3. Condition (4.7) is satisfied if and only if $a(\pi - x) \not\equiv (-1)^p a(x)$; see Lemma 7.1 of [26]. If (4.7) is violated, as for $f(u) = au^2$, then the right-hand side of equation (4.10) vanishes. In this case the correct 0th-order nontrivial bifurcation equation is not (4.10), and another "frequency–amplitude" relation is required, as in Section 2.4.

By the mountain pass theorem [7] (applied as in Chapter 2) there exists at least one nontrivial critical point of Φ_0, i.e., a solution of (4.10).

Due to the fact that the range equation can be solved by $w(\varepsilon, v)$ only for a Cantor-like set of parameters (ε, v), the solution of the bifurcation equation by means of variational methods is very difficult; see Remark 4.28 and [27]. Therefore we shall confine our attention here to the simplest situation, when at least one solution $\bar{v} \in V$ of (4.10) is nondegenerate in V, i.e.,

$$\ker \Phi''_{0|V}(\bar{v}) = \{0\} \tag{4.12}$$

(note that $\bar{v}_t \notin \ker \Phi''_{0|V}(\bar{v})$ because the functions of V are even in time). This nondegeneracy condition is somehow analogous to the "Arnold condition" in KAM theory.

[1] In the case $\exists v \in V$ such that $\int_\Omega a(x)v^{p+1} < 0$ we have to set $\omega^2 - 1 = 2\varepsilon$.

4.2 The Finite-Dimensional Reduction

To overcome the second difficulty (*ii*) we perform a further Lyapunov–Schmidt reduction to a *finite-dimensional* bifurcation equation on a subspace of V of dimension N depending only on $\bar{v}$ and the nonlinearity $a(x)u^p$.

Introducing the decomposition

$$V = V_1 \oplus V_2 ,$$

where

$$\begin{cases} V_1 := \left\{ v \in V \mid v = \sum_{l=1}^{N} \cos(lt) u_l \sin(lx) , \ u_l \in \mathbf{R} \right\}, \\ V_2 := \left\{ v \in V \mid v = \sum_{l \geq N+1} \cos(lt) u_l \sin(lx) , \ u_l \in \mathbf{R} \right\}, \end{cases}$$

and setting

$$v := v_1 + v_2 , \qquad v_1 \in V_1, \ v_2 \in V_2 ,$$

system (4.5) is equivalent to

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} f(v_1 + v_2 + w) , & (Q1) \\ -\Delta v_2 = \Pi_{V_2} f(v_1 + v_2 + w) , & (Q2) \\ L_\omega w = \varepsilon \Pi_W f(v_1 + v_2 + w) , & \textit{range equation} \end{cases} \tag{4.13}$$

where $\Pi_{V_i} \colon X_{\sigma,s} \mapsto V_i$ denotes the projector on V_i $(i = 1, 2)$.

Our strategy to find solutions of system (4.13) is the following:

Solution of the (Q2)-equation. We solve first the (Q2)-equation, obtaining $v_2 = v_2(v_1, w) \in V_2 \cap X_{\sigma,s+2}$ when $w \in W \cap X_{\sigma,s}$, provided we have chosen $N := \bar{N}$ large enough and $0 \leq \sigma \leq \bar{\sigma}$ small enough, depending only on $\bar{v}$ and the nonlinearity $a(x)u^p$.

Solution of the range equation. Next we solve the range equation, obtaining $w = w(\varepsilon, v_1) \in W \cap X_{\bar{\sigma}/2,s}$ by means of a Nash–Moser-type iteration scheme for (ε, v_1) belonging to some Cantor-like set B_∞.

Solution of the (Q1)-equation. Finally, we will solve the finite-dimensional (Q1)-equation in Section 4.4.

Remark 4.4. Under the nondegeneracy condition (4.12) we could also solve first the infinite-dimensional bifurcation equation in (4.5) and next the range equation. We prefer to proceed as below because this scheme allows us to deal also with degenerate cases; see [27].

4.2.1 Solution of the (Q2)-Equation

By the regularizing property of the Laplacian

$$(-\Delta)^{-1} \colon V \cap H^k(\Omega) \mapsto V \cap H^{k+2}(\Omega) , \qquad \forall k \geq 0 ,$$

and a direct bootstrap argument, the solution $\bar{v} \in V$ of equation (4.10) satisfies[2]

$$\bar{v} \in V \bigcap_{k \geq 0} H^k(\Omega) \equiv V \cap C^\infty(\Omega)\,.$$

In particular,

$$\|\bar{v}\|_{0,s} < \bar{R} \tag{4.14}$$

for some finite constant $\bar{R} > 0$.

The next lemma is the crucial step in overcoming the difficulty mentioned in (*ii*).

Proposition 4.5. (Solution of the ($Q2$)-equation) *There exists*

$$\bar{N} := N(\bar{R}, a(x), p) \in \mathbf{N}$$

such that

$$\forall\, 0 \leq \sigma \leq \bar{\sigma} := \frac{\ln 2}{\bar{N}}\,,\ \forall\, \|v_1\|_{0,s} \leq 2\bar{R}\,,\ \forall \|w\|_{\sigma,s} \leq 1\,,$$

there exists a unique

$$v_2 = v_2(v_1, w) \in V_2 \cap X_{\sigma,s+2}$$

with $\|v_2(v_1, w)\|_{\sigma,s} \leq 1$ *that solves the* ($Q2$)*-equation. The function* $v_2(\cdot,\cdot)$ *is* C^∞ *and* $D^k v_2$ *are bounded on bounded sets.*

Furthermore, the following "projection property" holds:

$$v_2(\Pi_{V_1}\bar{v}, 0) = \Pi_{V_2}\bar{v}\,. \tag{4.15}$$

If, in addition, $w \in X_{\sigma,s'}$ *for some* $s' \geq s$*, then* $v_2(v_1, w) \in X_{\sigma,s'+2}$ *and*

$$\|v_2(v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})\,. \tag{4.16}$$

Proof. Fixed points of the nonlinear operator $\mathcal{N}(v_1, w, \cdot)\colon V_2 \mapsto V_2$ defined by

$$\mathcal{N}(v_1, w, v_2) := (-\Delta)^{-1}\Pi_{V_2}\Big(a(x)(v_1 + w + v_2)^p\Big)$$

are solutions of the ($Q2$)-equation. Let

$$B := \Big\{v_2 \in V_2 \cap X_{\sigma,s} \mid \|v_2\|_{\sigma,s} \leq 1\Big\}\,.$$

We claim that there exists $\bar{N} := N(\bar{R}, a(x), p) \in \mathbf{N}$ such that $\forall 0 \leq \sigma \leq \bar{\sigma} := \ln 2/\bar{N}$, $\|v_1\|_{0,s} \leq 2\bar{R}$, $\|w\|_{\sigma,s} \leq 1$, the operator $\mathcal{N}(v_1, w, \cdot)$ is a contraction in B:

(i) $\|v_2\|_{\sigma,s} \leq 1 \Rightarrow \|\mathcal{N}(v_1, w, v_2)\|_{\sigma,s} \leq 1$;
(ii) $v_2, \widetilde{v}_2 \in B \Rightarrow \|\mathcal{N}(v_1, w, v_2) - \mathcal{N}(v_1, w, \widetilde{v}_2)\|_{\sigma,s} \leq \frac{1}{2}\|v_2 - \widetilde{v}_2\|_{\sigma,s}$.

[2] Even if $a(x) \in H^1((0,\pi), \mathbf{R})$ only, because the projection Π_V has a regularizing effect in the variable x.

Let us prove (i). $\forall u \in X_{\sigma,s}$,

$$\left\|(-\Delta)^{-1}\Pi_{V_2}u\right\|_{\sigma,s} \leq \frac{C}{(N+1)^2}\|u\|_{\sigma,s}\,,$$

and so $\forall \|w\|_{\sigma,s} \leq 1$, $\|v_1\|_{0,s} \leq 2\bar{R}$,

$$\begin{aligned}\left\|\mathcal{N}(v_1,w,v_2)\right\|_{\sigma,s} &\leq \frac{C}{(N+1)^2}\left\|a(x)(v_1+v_2+w)^p\right\|_{\sigma,s}\\ &\overset{(4.3)}{\leq} \frac{C'}{(N+1)^2}\Big(\|v_1\|^p_{\sigma,s}+\|v_2\|^p_{\sigma,s}+\|w\|^p_{\sigma,s}\Big)\\ &\leq \frac{C'}{(N+1)^2}\Big((4\bar{R})^p+\|v_2\|^p_{\sigma,s}+1\Big)\end{aligned} \tag{4.17}$$

for some $C' := C'(a(x),p)$, because

$$\|v_1\|_{\sigma,s} \leq \exp(\sigma N)\|v_1\|_{0,s} \leq 4\bar{R}$$

for all $0 \leq \sigma \leq (\ln 2/N)$.

Therefore, by (4.17), defining

$$\bar{N} := \sqrt{C'((4\bar{R})^p+2)}\,,$$

we have $\forall \|v_1\|_{0,s} \leq 2\bar{R}$, $\forall \|w\|_{\sigma,s} \leq 1$, $\forall 0 \leq \sigma \leq \bar{\sigma} := \ln 2/\bar{N}$,

$$\|v_2\|_{\sigma,s} \leq 1 \quad \Rightarrow \quad \left\|\mathcal{N}(v_1,w,v_2)\right\|_{\sigma,s} \leq \frac{C'}{(N+1)^2}\Big((4\bar{R})^p+1+1\Big) \leq 1\,,$$

and (i) follows. Property (ii) can be proved similarly, and the existence of a unique solution $v_2(v_1,w) \in B$ follows by the contraction mapping theorem.

Next $v_2(v_1,w) \in X_{\sigma,s+2}$ by a bootstrap argument using the regularization property of the Laplacian.

The C^∞ smoothness of $v_2(\cdot,\cdot)$ follows by the implicit function theorem and the C^∞ smoothness of $\mathcal{N}$; see [26] for details.

Let us prove (4.15). We may assume that $\bar{N}$ has been chosen so large that

$$\|\Pi_{V_2}\bar{v}\|_{0,s} \leq \frac{1}{2}\,.$$

Since $\bar{v}$ solves equation (4.10), $\Pi_{V_2}\bar{v}$ solves the $(Q2)$-equation associated with $(v_1,w) = (\Pi_{V_1}\bar{v},0)$ and $\|\Pi_{V_1}\bar{v}\|_{0,s} \leq \bar{R}$ by (4.14). Hence

$$\Pi_{V_2}\bar{v} = \mathcal{N}(\Pi_{V_1}\bar{v},0,\Pi_{V_2}\bar{v})$$

is a fixed point of $\mathcal{N}(\Pi_{V_1}\bar{v},0,\cdot)$ in B (for $\sigma = 0$), and by its uniqueness, $\Pi_{V_2}\bar{v} = v_2(\Pi_{V_1}\bar{v},0)$.

To prove (4.16) we exploit that $v_2(v_1,w) \in X_{\sigma,s}$ solves

$$v_2 = (-\Delta)^{-1}\Pi_{V_2}\Big(a(x)(v_1 + w + v_2)^p\Big).$$

Suppose that $w \in X_{\sigma,s'}$ for some $s' \geq s$. Then, by a direct bootstrap argument, using the regularizing properties of $(-\Delta)^{-1}$, the fact that $v_1 \in X_{\sigma,r}$, $\forall r \geq s$ (because V_1 is finite-dimensional), the Banach algebra property of $X_{\sigma,r}$, we derive that $v_2(v_1, w) \in X_{\sigma,s'+2}$ and $\|v_2(v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})$. ■

Remark 4.6. Proposition 4.5 solves the difficulty mentioned in (*ii*) because $v_2(v_1, w)$ has always the same smoothness of $w \in X_{\sigma,s}$ (actually it also gains two derivatives, $w \in X_{\sigma,s+2}$). Therefore during the Nash–Moser iteration the "tail" $v_2(v_1, w_n)$ will "adjust" its smoothness like the iterated w_n (which decreases its analyticity).

Remark 4.7. We perform the finite-dimensional reduction in $B(2\bar{R}; V_1) := \{v_1 \in V_1 \mid \|v_1\|_{0,s} \leq 2\bar{R}\}$ because we expect solutions of the $(Q1)$-equation close to $\Pi_{V_1}\bar{v}$ and $\|\Pi_{V_1}\bar{v}\|_{0,s} \leq \bar{R}$. This will be observed in Section 4.4.

We stress that in the sequel we shall consider as *fixed* the constants $\bar{N}$ and $\bar{\sigma}$, which depend only on $\bar{v}$ and the nonlinearity $a(x)u^p$.

4.3 Solution of the Range Equation

By the previous section we are reduced to solving the range equation with $v_2 = v_2(v_1, w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(v_1, w), \tag{4.18}$$

where

$$\Gamma(v_1, w) := f\Big(v_1 + w + v_2(v_1, w)\Big). \tag{4.19}$$

The solution $w = w(\varepsilon, v_1)$ of the range equation (4.18) will be obtained by means of a Nash–Moser iteration scheme for (ε, v_1) belonging to a Cantor-like set of parameters.

Theorem 4.8. (Solution of the range equation) *For $\varepsilon_0 > 0$ small enough, there exist*

$$\widetilde{w}(\cdot,\cdot) \in C^\infty\Big(A_0, W \cap X_{\bar{\sigma}/2,s}\Big),$$

where $A_0 := [0, \varepsilon_0] \times B(2\bar{R}; V_1)$, satisfying

$$\Big\|\widetilde{w}(\varepsilon, v_1)\Big\|_{\bar{\sigma}/2,s} \leq C\frac{\varepsilon}{\gamma}, \quad \Big\|D^k\widetilde{w}(\varepsilon, v_1)\Big\|_{\bar{\sigma}/2,s} \leq C(k), \tag{4.20}$$

and the "large" Cantor set $B_\infty \subset A_0$ defined below, such that

$$\forall(\varepsilon, v_1) \in B_\infty, \quad \widetilde{w}(\varepsilon, v_1) \text{ solves the range equation (4.18)}.$$

The Cantor set B_∞ can be written explicitly as

$$B_\infty := \Big\{ (\varepsilon, v_1) \in A_0 \,:\, \Big|\omega(\varepsilon)l - j - \varepsilon \frac{M(v_1, \widetilde{w}(\varepsilon, v_1))}{2j}\Big| \geq \frac{2\gamma}{(l+j)^\tau}, \\ \Big|\omega(\varepsilon)l - j\Big| \geq \frac{2\gamma}{(l+j)^\tau}, \ \forall j \geq 1, \ \forall l \geq \frac{1}{3\varepsilon}, \ l \neq j \Big\} \subset A_0,$$

where $\omega(\varepsilon) = \sqrt{1 - 2\varepsilon}$ *and*

$$M(v_1, w) := \frac{1}{|\Omega|} \int_\Omega (\partial_u f)\Big(x, v_1 + w + v_2(v_1, w)\Big)\, \mathrm{d}x \mathrm{d}t. \tag{4.21}$$

As explained in the introduction, to get the invertibility of the linearized operators we have to excise values of the parameters (ε, v_1). For this reason, it is more convenient to apply a slight modification of the analytic Nash–Moser scheme described in Section 3.2, introducing, as in Section 3.3, also a sequence of "smoothing" operators via finite-dimensional Fourier truncations, as in Remark 3.5.

The advantage is that we avoid having to invert the infinite-dimensional operator $\mathcal{L}(\varepsilon, v, w)$, replacing it by a sequence of *finite*-dimensional matrices of *increasing dimensions*, imposing at each step a finite number of conditions on (ε, v_1); see property (P3) below.

We consider the orthogonal splitting

$$W = W^{(n)} \oplus W^{(n)\perp},$$

where

$$W^{(n)} = \Big\{ w \in W \;\Big|\; w = \sum_{|l| \leq L_n} \exp(\mathrm{i}lt)\, w_l(x) \Big\}, \tag{4.22}$$

$$W^{(n)\perp} = \Big\{ w \in W \;\Big|\; w = \sum_{|l| > L_n} \exp(\mathrm{i}lt)\, w_l(x) \Big\}, \tag{4.23}$$

and

$$L_n := L_0 2^n$$

with $L_0 \in \mathbf{N}$ large enough. We denote by

$$P_n \colon W \to W^{(n)} \qquad \text{and} \qquad P_n^\perp \colon W \to W^{(n)\perp}$$

the orthogonal projectors onto $W^{(n)}$ and $W^{(n)\perp}$.

The existence of a solution of the range equation is based on the following properties (P1)–(P3).

- **(P1)** (**Regularity**) $\Gamma(\cdot,\cdot) \in C^\infty\Big(B(2\bar{R}; V_1) \times B(1; W \cap X_{\sigma,s}), X_{\sigma,s}\Big)$. Moreover, $D^k\Gamma$ are bounded on $B(2\bar{R}; V_1) \times B(1; W \cap X_{\sigma,s})$.

(P1) is a consequence of the C^∞-regularity of the Nemitsky operator induced by $f(u)$ on $X_{\sigma,s}$ and of the C^∞-regularity of the map $v_2(\cdot,\cdot)$ proved in Proposition 4.5.

- **(P2) (Smoothing estimate)** $\forall\, w \in W^{(n)\perp} \cap X_{\sigma,s}$ and $\forall\, 0 \leq \sigma' \leq \sigma$, $\|w\|_{\sigma',s} \leq \exp(-L_n(\sigma - \sigma'))\|w\|_{\sigma,s}$.

  The standard property (P2) follows from

$$\begin{aligned}\|w\|^2_{\sigma',s} &= \sum_{|l|>L_n} \exp(2\sigma'|l|)(l^{2s}+1)\|w_l\|^2_{H^1} \\ &= \sum_{|l|>L_n} \exp(-2(\sigma-\sigma')|l|)\exp(2\sigma|l|)(l^{2s}+1)\|w_l\|^2_{H^1} \\ &\leq \exp(-2(\sigma-\sigma')L_n)\|w\|^2_{\sigma,s}\,.\end{aligned}$$

  The next property (P3) is the *invertibility property* of the linearized operator

$$\mathcal{L}_n(\varepsilon, v_1, w)\,[h] := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(v_1, w)\,[h]\,, \tag{4.24}$$

$\forall h \in W^{(n)}$.

- **(P3) (Invertibility of $\mathcal{L}_n$)** Let $\gamma \in (0,1)$, $\tau \in (1,2)$. Suppose $\exists\, C_1 > 0$ such that

$$\left\|(\partial_u f)(v_1 + w + v_2(v_1, w))\right\|_{\sigma, s+\frac{2\tau(\tau-1)}{2-\tau}} \leq C_1\,. \tag{4.25}$$

  For $\varepsilon_0 > 0$ small enough, if (ε, v_1) belongs to

$$\begin{aligned}\mathcal{A}_n(w) := \Big\{&(\varepsilon, v_1) \in [0,\varepsilon_0] \times B(2\bar{R}, V_1) \mid |\omega(\varepsilon)l - j| \geq \frac{\gamma}{(l+j)^\tau}\,, \\ &\left|\omega(\varepsilon)l - j - \varepsilon\frac{M(v_1,w)}{2j}\right| \geq \frac{\gamma}{(l+j)^\tau}\,,\ \forall (j,l) \in \mathbf{N}\times\mathbf{N} \\ &l \neq j,\ \frac{1}{3\varepsilon} < l,\ l \leq L_n,\ j \leq 2L_n\Big\}\end{aligned} \tag{4.26}$$

  (first-order Melnikov nonresonance conditions), then $\mathcal{L}_n(\varepsilon, v_1, w)$ is invertible on $W^{(n)}$ and

$$\left\|\mathcal{L}_n^{-1}(\varepsilon, v_1, w)\,[h]\right\|_{\sigma,s} \leq \frac{K}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s} \tag{4.27}$$

  for some $K > 0$ (independent of n).

Remark 4.9. An estimate like (4.27) that holds for any $n \geq 0$ is tantamount to saying that the inverse of the infinite-dimensional operator $\mathcal{L}(\varepsilon, v_1, w)\,[h] := L_\omega h - \varepsilon \Pi_W D_w \Gamma(v_1, w)\,[h]$ satisfies $\|\mathcal{L}^{-1}(\varepsilon, v_1, w)\,[h]\|_{\sigma,s} \leq K'\gamma^{-1}\|h\|_{\sigma,s+\tau-1}$, i.e., it loses $(\tau - 1)$ derivatives. This can be seen via a "dyadic decomposition" using that $L_n = L_0 2^n$, $L_{n+1} = 2L_n$.

Property (P3) is the real core of the convergence proof and where the analysis of the small divisors enters into play. Property (P3) is proved in Section 4.5.

4.3.1 The Nash–Moser Scheme

Following the proof of the "analytic" Nash–Moser theorem, Theorem 3.1, we first define the sequence

$$\sigma_0 := \bar{\sigma} , \qquad \sigma_{n+1} = \sigma_n - \gamma_n , \qquad \forall\, n \geq 0 , \tag{4.28}$$

where

$$\gamma_n := \frac{\gamma_0}{n^2 + 1}$$

denotes the "loss of analyticity" at each step of the iteration, and we choose $\gamma_0 > 0$ small such that the "total loss of analyticity" satisfies

$$\sum_{n \geq 0} \gamma_n = \sum_{n \geq 0} \frac{\gamma_0}{n^2 + 1} \leq \frac{\bar{\sigma}}{2} . \tag{4.29}$$

By (4.28) and (4.29),

$$\sigma_n > \frac{\bar{\sigma}}{2} > 0 , \quad \forall n .$$

Proposition 4.10. (Induction) $\exists\, L_0 := L_0(\gamma, \tau) > 0$, $\varepsilon_0 := \varepsilon_0(\gamma, \tau) > 0$, *such that* $\forall \varepsilon \gamma^{-1} < \varepsilon_0$, $\forall n \geq 0$ *there exists a classical solution* $w_n = w_n(\varepsilon, v_1) \in W^{(n)}$ *of the equation*

$$(P_n) \qquad\qquad L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(v_1, w_n) = 0$$

defined inductively for

$$(\varepsilon, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0 := [0, \varepsilon_0] \times B(2\bar{R}; V_1) ,$$

where

$$A_n := A_{n-1} \cap \mathcal{A}_n(w_{n-1}) \neq \emptyset \tag{4.30}$$

and the set $\mathcal{A}_n(w_{n-1})$ *is defined by (4.26).*

Furthermore $w_n = \sum_{i=0}^{n} h_i$, *with* $h_i \in W^{(i)}$ *satisfying*

$$\|h_i\|_{\sigma_i, s} \leq C \frac{\varepsilon}{\gamma} \exp(-\chi^i) , \quad \chi = \frac{3}{2} , \tag{4.31}$$

$w_n(\cdot, \cdot) \in C^\infty(A_n, W^{(n)})$, *and*

$$\Big\| w_n(\varepsilon, v_1) \Big\|_{\sigma_n, s} \leq \frac{\varepsilon}{\gamma} K_1 , \quad \Big\| D^k w_n(\varepsilon, v_1) \Big\|_{\sigma_n, s} \leq K_1(k) \tag{4.32}$$

for any $k \geq 0$. *Furthermore, there exists* $\widetilde{w}_n(\cdot, \cdot) \in C^\infty\,(A_0, W^{(n)})$ *such that*

$$\widetilde{w}_n(\varepsilon, v_1) = w_n(\varepsilon, v_1) , \quad \forall (\varepsilon, v_1) \in \bigcap_{i=0}^{n} \widetilde{A}_i , \tag{4.33}$$

where for some $\nu := \nu(\gamma, \tau) > 0$ *small enough,*

$$\widetilde{A}_i := \left\{ (\varepsilon, v_1) \in A_i \mid \operatorname{dist}((\varepsilon, v_1), \partial A_i) \geq \frac{2\nu}{L_i^3} \right\} \subset A_i$$

$\forall i = 0, \ldots, n$. *The function* $\widetilde{w}_n$ *satisfies*

$$\left\| \widetilde{w}_n(\varepsilon, v_1) \right\|_{\sigma_n, s} \leq \frac{\varepsilon}{\gamma} C \,, \qquad \left\| D^k \widetilde{w}_n(\varepsilon, v_1) \right\|_{\sigma_n, s} \leq C(k) \,, \tag{4.34}$$

for any $k \geq 0$.

Proof. The proof proceeds by induction.

First step: initialization. Let L_0 be given. For any

$$|\omega - 1| L_0 \leq \frac{1}{2} \,,$$

then $L_{\omega | W^{(0)}}$ is invertible and

$$\| L_\omega^{-1} h \|_{\sigma_0, s} \leq 2 \| h \|_{\sigma_0, s} \,, \quad \forall h \in W^{(0)}$$

(this is the finite-dimensional situation as in Chapter 1). Indeed, the eigenvalues of $L_{\omega | W^{(0)}}$ are $-\omega^2 l^2 + j^2$, $\forall\, 0 \leq l \leq L_0$, $j \geq 1$, $j \neq l$, and

$$\begin{aligned} | -\omega^2 l^2 + j^2 | &= | -\omega l + j | (\omega l + j) \\ &\geq \Big(|j - l| - |\omega - 1| L_0 \Big) (\omega l + j) \geq 1 - \frac{1}{2} \,. \end{aligned}$$

Therefore, using Property (P1), $\forall \varepsilon \gamma^{-1} < \varepsilon_1(\gamma, L_0)$ small enough, $\forall v_1 \in B(2\bar{R}; V_1)$, equation (P_0) has a unique solution $w_0(\varepsilon, v_1) \in W^{(0)}$ satisfying

$$\| w_0(\varepsilon, v_1) \|_{\sigma_0, s} \leq K_0 \varepsilon \,.$$

Moreover $w_0(\cdot, \cdot) \in C^\infty(A_0, W^{(0)})$ and $\| D^k w_0(\varepsilon, v_1) \|_{\sigma_0, s} \leq C(k)$.

Second step: iteration. Suppose we have already defined a solution $w_n \in W^{(n)}$ of equation (P_n) satisfying the properties stated in the proposition. We want to define

$$w_{n+1} = w_n + h_{n+1} \,, \qquad h_{n+1} \in W^{(n+1)}$$

as an *exact* solution of the equation

$$(P_{n+1}) \qquad L_\omega w_{n+1} - \varepsilon P_{n+1} \Pi_W \Gamma(v_1, w_{n+1}) = 0 \,.$$

For $h \in W^{(n+1)}$ we develop

$$\begin{aligned} L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi_W \Gamma(v_1, w_n + h) = {} & L_\omega w_n - \varepsilon P_{n+1} \Pi_W \Gamma(v_1, w_n) \\ & + L_\omega h - \varepsilon P_{n+1} \Pi_W D_w \Gamma(v_1, w_n)\,[h] \\ & + R(h) \\ = {} & r_n + \mathcal{L}_{n+1}(\varepsilon, v_1, w_n)\,[h] + R(h) \,, \end{aligned}$$

where, since w_n solves equation (P_n),

$$\begin{cases} r_n := L_\omega w_n - \varepsilon P_{n+1}\Pi_W \Gamma(v_1, w_n) = -\varepsilon P_n^\perp P_{n+1}\Pi_W \Gamma(v_1, w_n)\,, \\ R(h) := -\varepsilon P_{n+1}\Pi_W\Big(\Gamma(v_1, w_n + h) - \Gamma(v_1, w_n) - D_w \Gamma(v_1, w_n)\,[h]\Big)\,. \end{cases}$$

Hence equation (P_{n+1}) amounts to solving

$$\mathcal{L}_{n+1}(\varepsilon, v_1, w_n)\,[h] = -r_n - R(h)\,. \tag{4.35}$$

The remainder r_n is "superexponentially" small because

$$\begin{aligned} \|r_n\|_{\sigma_{n+1},s} &\overset{(P2)}{\leq} \varepsilon\, C \exp(-L_n \gamma_n) \Big\| P_{n+1}\Pi_W \Gamma(v_1, w_n)\Big\|_{\sigma_n,s} \\ &\leq \varepsilon\, C' \exp(-L_n \gamma_n) \Big\| \Gamma(v_1, w_n)\Big\|_{\sigma_n,s} \\ &\overset{(P1)}{\leq} \varepsilon\, C'' \exp(-L_n \gamma_n) \end{aligned} \tag{4.36}$$

using also (4.32).

The term $R(h)$ is "quadratic" in h, since by (P1) and (4.32),

$$\begin{cases} \|R(h)\|_{\sigma_{n+1},s} \leq C\varepsilon\, \|h\|^2_{\sigma_{n+1},s}\,, \\ \|R(h) - R(h')\|_{\sigma_{n+1},s} \leq C\varepsilon\, (\|h\|_{\sigma_{n+1},s} + \|h'\|_{\sigma_{n+1},s})\, \|h - h'\|_{\sigma_{n+1},s}\,, \end{cases} \tag{4.37}$$

for all $\|h\|_{\sigma_{n+1},s}$, $\|h'\|_{\sigma_{n+1},s}$ small enough.

Lemma 4.11. *There is $C_1 > 0$ such that $\forall \|v_1\|_{0,s} \leq 2\bar{R}$, $\forall n \geq 0$,*

$$\Big\|(\partial_u f)\Big(v_1 + w_n + v_2(v_1, w_n)\Big)\Big\|_{\sigma_{n+1},s+\frac{2\tau(\tau-1)}{2-\tau}} \leq C_1\,. \tag{4.38}$$

Proof. Since $w_n = \sum_{i=0}^n h_i$,

$$\begin{aligned} \|w_n\|_{\sigma_{n+1},s+\frac{2\tau(\tau-1)}{2-\tau}} &\leq \sum_{i=0}^n \|h_i\|_{\sigma_i,s+\frac{2\tau(\tau-1)}{2-\tau}} \\ &\overset{(*)}{\leq} \sum_{i=0}^n \frac{C(\tau)\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma_{n+1})^{\frac{2\tau(\tau-1)}{2-\tau}}} \\ &\overset{(4.31)}{\leq} C\frac{\varepsilon}{\gamma} \sum_{i=0}^n \frac{\exp(-\chi^i)}{\gamma_i^{2\tau(\tau-1)/(2-\tau)}} \\ &\leq C\frac{\varepsilon}{\gamma} \sum_{i=0}^{+\infty} \exp(-\chi^i)\Big(\frac{1+i^2}{\gamma_0}\Big)^{\frac{2\tau(\tau-1)}{2-\tau}} = \frac{\varepsilon}{\gamma}K\,, \end{aligned}$$

since $\sigma_i - \sigma_{n+1} \geq \gamma_i := \gamma_0/(1+i^2)$, $\forall i = 0, \ldots, n$ and using in $(*)$ the elementary inequality

$$\max_{k\geq 1} k^\alpha \exp\{-(\sigma_i - \sigma_{n+1})k\} \leq \frac{C(\alpha)}{(\sigma_i - \sigma_{n+1})^\alpha}.$$

By (4.16), also

$$\|v_2(v_1, w_n)\|_{\sigma_{n+1}, s+\frac{2\tau(\tau-1)}{2-\tau}} \leq K',$$

and (4.38) holds by the Banach algebra property of $X_{\sigma, s+2\tau(\tau-1)/(2-\tau)}$. ■

By the previous lemma, for

$$(\varepsilon, v_1) \in A_{n+1} := A_n \cap \mathcal{A}_{n+1}(w_n), \tag{4.39}$$

property (P3) applies: the linear operator $\mathcal{L}_{n+1}(\varepsilon, v_1, w_n)$ is invertible and

$$\left\| \mathcal{L}_{n+1}(\varepsilon, v_1, w_n)^{-1} \right\|_{\sigma_{n+1}, s} \leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}, \qquad \forall (\varepsilon, v_1) \in A_{n+1}. \tag{4.40}$$

Remark 4.12. Note that $A_n \equiv \cdots \equiv A_1 \equiv A_0$ if $L_0 2^n < \frac{1}{3\varepsilon}$.

Remark 4.13. To show that $A_\infty := \cap_{n\geq 0} A_n$ is not empty but is actually a "large" set, the key ingredient to exploit is again the "superexponential" convergence (4.31) of the approximate solutions w_n. The method we choose here is the following: we define a map $\widetilde{w}(\cdot, \cdot)$ on the *whole* set A_0 (see (4.60)) and a set $B_\infty \subset A_\infty$ depending only on $\widetilde{w}$; see Lemma 4.19. In Proposition 4.24 we prove that B_∞, and so A_∞, is a "large" set. An advantage of this approach is the very explicit expression of the Cantor set B_∞. This is especially exploited in the fine-measure estimate of [27].

By (4.35), equation (P_{n+1}) is equivalent to finding a fixed point

$$h = \mathcal{G}(\varepsilon, v_1, w_n, h), \qquad h \in W^{(n+1)}, \tag{4.41}$$

of the nonlinear operator

$$\mathcal{G}(\varepsilon, v_1, w_n, h) := -\mathcal{L}_{n+1}(\varepsilon, v_1, w_n)^{-1}\Big(r_n + R(h)\Big).$$

Lemma 4.14. (Contraction) *There exist $L_0(\gamma, \tau) > 0$, $\varepsilon_0(L_0, \gamma, \tau) > 0$, such that $\forall \varepsilon\gamma^{-1} < \varepsilon_0$, the operator $\mathcal{G}(\varepsilon, v_1, w_n, \cdot)$ is, for any $n \geq 0$, a contraction in the ball*

$$B(\rho_{n+1}; W^{(n+1)}) := \Big\{ h \in W^{(n+1)} \mid \|h\|_{\sigma_{n+1}, s} \leq \rho_{n+1} := \frac{\varepsilon}{\gamma} \exp\left(-\chi^{n+1}\right) \Big\}.$$

Proof. We first prove that $\mathcal{G}(\varepsilon, v_1, w_n, \cdot)$ maps the ball $B(\rho_{n+1}; W^{(n+1)})$ into itself. We have

$$\begin{aligned} \left\| \mathcal{G}(\varepsilon, v_1, w_n, h) \right\|_{\sigma_{n+1}, s} &= \left\| \mathcal{L}_{n+1}(\varepsilon, v_1, w_n)^{-1}\Big(r_n + R(h)\Big) \right\|_{\sigma_{n+1}, s} \\ &\overset{(4.40)}{\leq} \frac{C}{\gamma}(L_{n+1})^{\tau-1}\Big(\|r_n\|_{\sigma_{n+1}, s} + \|R(h)\|_{\sigma_{n+1}, s} \Big) \\ &\leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}\Big(\varepsilon \exp\left(-L_n \gamma_n\right) + \varepsilon \|h\|^2_{\sigma_{n+1}, s} \Big) \end{aligned}$$

by (4.36) and (4.37). Therefore, if $\|h\|_{\sigma_{n+1},s} \le \rho_{n+1}$, then

$$\left\| \mathcal{G}(\varepsilon, v_1, w_n, h) \right\|_{\sigma_{n+1},s} \le \frac{C'}{\gamma} (L_{n+1})^{\tau-1} \varepsilon \Big(\exp(-L_n \gamma_n) + \rho_{n+1}^2 \Big) \le \rho_{n+1} \,,$$

provided that

$$C' \frac{\varepsilon}{\gamma} (L_{n+1})^{\tau-1} \exp(-L_n \gamma_n) \le \frac{\rho_{n+1}}{2} \tag{4.42}$$

and

$$C' \frac{\varepsilon}{\gamma} (L_{n+1})^{\tau-1} \rho_{n+1} \le \frac{1}{2} \,. \tag{4.43}$$

Then (4.42) becomes, for $\rho_{n+1} := \varepsilon \gamma^{-1} \exp(-\chi^{n+1})$,

$$C' (L_{n+1})^{\tau-1} \exp(-L_n \gamma_n) \le \frac{1}{2} \exp(-\chi^{n+1}) \,,$$

which for $L_n := L_0 2^n$, $\gamma_n := \gamma_0/(1+n^2)$, and $L_0 := L_0(\gamma, \tau) > 0$ large enough is satisfied $\forall n \ge 0$. And (4.43) becomes

$$C' \frac{\varepsilon^2}{\gamma^2} \Big(L_0(\gamma, \tau) 2^{n+1} \Big)^{\tau-1} \exp(-\chi^{n+1}) \le \frac{1}{2} \,,$$

which is satisfied for $\varepsilon \gamma^{-1} \le \varepsilon_0(L_0, \gamma, \tau)$ small enough, $\forall n \ge 0$.

With similar estimates, by (4.37), we get $\forall h, h' \in B(\rho_{n+1}; W^{(n+1)})$,

$$\left\| \mathcal{G}(\varepsilon, v_1, w_n, h') - \mathcal{G}(\varepsilon, v_1, w_n, h) \right\|_{\sigma_{n+1},s} \le \frac{1}{2} \|h - h'\|_{\sigma_{n+1},s} \,,$$

again for L_0 large enough and $\varepsilon \gamma^{-1} \le \varepsilon_0(L_0, \gamma, \tau)$ small enough, uniformly in n. ■

By the contraction mapping theorem we deduce the existence of a unique $h_{n+1} \in W^{(n+1)}$ solving (4.41) and satisfying

$$\left\| h_{n+1} \right\|_{\sigma_{n+1},s} \le \rho_{n+1} = \frac{\varepsilon}{\gamma} \exp(-\chi^{n+1}) \,. \tag{4.44}$$

Furthermore, by an elementary bootstrap in (4.35), since $\mathcal{L}_{n+1}^{-1}(\varepsilon, v_1, w_n)$ gains two spatial derivatives, each $h_n(t, \cdot)$ is in $H^3(0, \pi)$. Hence w_n is a classical solution of equation (P_n). This completes the existence proof.

We also note that by (4.31) and (4.35),

$$\|\partial_{tt} h_n\|_{\sigma_n,s} \,,\ \|\partial_{xx} h_n\|_{\sigma_n,s} \le K \varepsilon \exp(-\chi_*^n) \tag{4.45}$$

for some $1 < \chi_* < \chi$, $K > 0$.

Finally, $w_{n+1} = \sum_{i=0}^{n+1} h_i$, $h_i \in W^{(i)}$, satisfies

$$\|w_{n+1}\|_{\sigma_{n+1},s} \le \sum_{i=0}^{n+1} \|h_i\|_{\sigma_i,s} \le \sum_{i=0}^{+\infty} C \frac{\varepsilon}{\gamma} \exp(-\chi^i) = K_1 \frac{\varepsilon}{\gamma} \,,$$

and the left estimate of (4.32) holds.

Remark 4.15. A difference with respect to the "quadratic" Nash–Moser (plus the truncation) scheme in [51] is that $h_n(\varepsilon, v_1)$ is found as an exact solution of equation (P_n), and not just as a solution of the linearized equation $r_n + \mathcal{L}_{n+1}(\varepsilon, v_1, w_n)[h] = 0$. This is convenient in proving the regularity of $h_n(\cdot,\cdot)$ in the next lemma.

Lemma 4.16. (Estimates for the derivatives) *$h_n(\cdot,\cdot) \in C^\infty(A_n, W^{(n)})$, $\forall n \geq 0$, and*

$$\left\| D^k h_n(\varepsilon, v_1) \right\|_{\sigma_n, s} \leq [K_1(k, \bar{\chi})]^{n+1} \exp(-\bar{\chi}^n) \tag{4.46}$$

for some $\bar{\chi} \in (1, \chi)$.

As a consequence, $w_n(\cdot,\cdot) \in C^\infty(A_n, W^{(n)})$ and (4.32) holds.

Proof. By the first step in the proof of Proposition 4.10, $h_0 = w_0 \in C^\infty(A_0, W^{(0)})$ and $\|D^k w_0(\varepsilon, v_1)\|_{\sigma_0, s} \leq C(k)$.

Next, assume by induction that $h_n(\cdot,\cdot) \in C^\infty(A_n, W^{(n)})$. We shall prove that $h_{n+1}(\cdot,\cdot) \in C^\infty(A_{n+1}, W^{(n+1)})$.

We have that $h_{n+1}(\varepsilon, v_1)$ is a solution of

$$(P_{n+1}) \qquad U_{n+1}\Big(\varepsilon, v_1, h_{n+1}(\varepsilon, v_1)\Big) = 0\,,$$

where

$$U_{n+1}(\varepsilon, v_1, h) := L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi_W \Gamma(v_1, w_n + h)\,.$$

We claim that $D_h U_{n+1}(\varepsilon, v_1, h_{n+1}) = \mathcal{L}_{n+1}(\varepsilon, v_1, w_{n+1})$ ($= \mathcal{L}_{n+1}(w_{n+1})$ for short) is invertible and

$$\left\| \Big(D_h U_{n+1}(\varepsilon, v_1, h_{n+1}) \Big)^{-1} \right\|_{\sigma_{n+1}, s} \leq \frac{C'}{\gamma} (L_{n+1})^{\tau - 1}\,. \tag{4.47}$$

By the implicit function theorem, $h_{n+1}(\cdot,\cdot) \in C^\infty(A_{n+1}, W^{(n+1)})$. To prove (4.47), we first recall that $\mathcal{L}_{n+1}(w_n)$ is invertible and (4.40) holds. We have

$$\begin{aligned}
&\left\| \mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n) \right\|_{\sigma_{n+1}, s} \\
&\quad = \left\| \varepsilon P_{n+1} \Pi_W \Big(D_w \Gamma(v_1, w_{n+1}) - D_w \Gamma(v_1, w_n) \Big) \right\|_{\sigma_{n+1}, s} \\
&\quad \overset{(P1)}{\leq} C\varepsilon \, \|h_{n+1}\|_{\sigma_{n+1}, s} \\
&\quad \overset{(4.44)}{\leq} C \frac{\varepsilon^2}{\gamma} \exp(-\chi^{n+1})\,.
\end{aligned} \tag{4.48}$$

Therefore

$$\mathcal{L}_{n+1}(w_{n+1}) = \mathcal{L}_{n+1}(w_n) \Big[\mathrm{Id} + \mathcal{L}_{n+1}(w_n)^{-1} \Big(\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n) \Big) \Big] \tag{4.49}$$

is invertible whenever (recall (4.40) and (4.48))

$$\left\| \mathcal{L}_{n+1}(w_n)^{-1}\Big(\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)\Big) \right\|_{\sigma_{n+1},s} \leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1}) < \frac{1}{2}, \tag{4.50}$$

provided that $\varepsilon\gamma^{-1}$ is small enough (note that $(L_{n+1})^{\tau-1} = (L_0 2^{n+1})^{\tau-1} \ll \exp(\chi^{n+1})$ for n large). Furthermore, by (4.49), (4.40), (4.50), estimate (4.47) holds.

We now prove in detail estimate (4.46) for $k = 1$. Let us define $\lambda := (\varepsilon, v_1) \in A_{n+1}$. Differentiating equation (P_{n+1}) with respect to λ, we obtain

$$(P'_{n+1}) \qquad \mathcal{L}_{n+1}(\varepsilon, v_1, w_{n+1})\Big[\partial_\lambda h_{n+1}(\varepsilon, v_1)\Big] = -(\partial_\lambda U_{n+1})\Big(\varepsilon, v_1, h_{n+1}(\varepsilon, v_1)\Big)$$

and therefore

$$\left\| \partial_\lambda h_{n+1} \right\|_{\sigma_{n+1},s} \overset{(4.47)}{\leq} \frac{C}{\gamma}(L_{n+1})^{\tau-1}\left\| (\partial_\lambda U_{n+1})(\varepsilon, v_1, h_{n+1}) \right\|_{\sigma_{n+1},s}. \tag{4.51}$$

Since w_n solves equation (P_n), that is, $L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(v_1, w_n)$,

$$U_{n+1}(\varepsilon, v_1, h) = L_\omega h + \varepsilon(P_n \Pi_W \Gamma(v_1, w_n) - P_{n+1}\Pi_W \Gamma(v_1, w_n + h))$$

and we can write

$$(\partial_\lambda U_{n+1})(\varepsilon, v_1, h) = (\partial_\lambda U_{n+1})(\varepsilon, v_1, 0) + r(\varepsilon, v_1, h), \tag{4.52}$$

where

$$\begin{aligned}(\partial_\lambda U_{n+1})(\varepsilon, v_1, 0) &= (P_n - P_{n+1})\Pi_W \partial_\lambda\left[\varepsilon\Gamma(v_1, w_n(\varepsilon, v_1))\right] \\ &= -\varepsilon P_n^\perp P_{n+1}\Pi_W\Big[(\partial_\lambda\Gamma)(v_1, w_n) + (\partial_w\Gamma)(v_1, w_n)\left[\partial_\lambda w_n\right]\Big] \\ &\quad -\partial_\lambda(\varepsilon) P_n^\perp P_{n+1}\Gamma(v_1, w_n)\end{aligned} \tag{4.53}$$

and

$$\begin{aligned} r(\varepsilon, v_1, h) &:= \varepsilon P_{n+1}\Pi_W\Big[(\partial_\lambda\Gamma)(v_1, w_n) - (\partial_\lambda\Gamma)(v_1, w_n + h)\Big] \\ &\quad + \varepsilon P_{n+1}\Pi_W\Big[(\partial_w\Gamma)(v_1, w_n) - (\partial_w\Gamma)(v_1, w_n + h)\Big]\left[\partial_\lambda w_n\right] \\ &\quad + \partial_\lambda(L_{\omega(\varepsilon)}h) + \partial_\lambda(\varepsilon)\, P_{n+1}\Pi_W(\Gamma(v_1, w_n) - \Gamma(v_1, w_n + h))\end{aligned}$$

with $\partial_\lambda(L_{\omega(\varepsilon)}h) = 0$, $\partial_\lambda(\varepsilon) = 0$ if $\lambda \neq \varepsilon$ and $\partial_\varepsilon(L_{\omega(\varepsilon)}h) = -2h_{tt}$ (recall that $\omega^2 = 1 - 2\varepsilon$). Therefore

$$\left\| r(\varepsilon, v_1, h) \right\|_{\sigma_{n+1},s} \leq C\varepsilon\, \|h\|_{\sigma_{n+1},s}\Big(1 + \|\partial_\lambda w_n\|_{\sigma_{n+1},s}\Big) + CL_{n+1}^2\|h\|_{\sigma_{n+1},s}. \tag{4.54}$$

We now estimate $(\partial_\lambda U_{n+1})(\varepsilon, v_1, 0)$. By (4.53),

$$\Big\|(\partial_\lambda U_{n+1})(\varepsilon, v_1, 0)\Big\|_{\sigma_{n+1},s} \overset{(P2)}{\leq} \exp(-L_n\gamma_n)\Big[\varepsilon\Big\|(\partial_\lambda\Gamma)(v_1, w_n) + (\partial_w\Gamma)(v_1, w_n)\,[\partial_\lambda w_n]\Big\|_{\sigma_n,s} + \Big\|\Gamma(v_1, w_n)\Big\|_{\sigma_n,s}\Big]$$
$$\overset{(P1)}{\leq} C\exp(-L_n\gamma_n)\Big(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}\Big)\,. \tag{4.55}$$

Combining (4.51), (4.52), (4.54), (4.55), and (4.44) yields

$$\Big\|\partial_\lambda h_{n+1}\Big\|_{\sigma_{n+1},s} \leq \frac{C}{\gamma}(L_{n+1})^{\tau+1}\Big(\frac{\varepsilon}{\gamma}\exp(-\chi^{n+1}) + \exp(-L_n\gamma_n)\Big) \times\Big(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}\Big)$$
$$\leq C(\bar\chi)\;\exp(-\bar\chi^{n+1})\Big(1 + \sum_{i=0}^{n}\|\partial_\lambda h_i\|_{\sigma_i,s}\Big) \tag{4.56}$$

for any $\bar\chi \in (1, \chi)$. Reasoning by induction (4.56) implies (4.46) for $k = 1$. ■

We now define, by interpolation, say, a C^∞-extension $\widetilde{w}_n(\varepsilon, v_1)$ of $w_n(\varepsilon, v_1)$.

Lemma 4.17. (Whitney C^∞ extension) *Let*

$$\widetilde{A}_i := \Big\{(\varepsilon, v_1) \in A_i \mid \operatorname{dist}((\varepsilon, v_1), \partial A_i) \geq \frac{2\nu}{L_i^3}\Big\} \subset A_i\,,$$

where $\nu := \nu(\gamma, \tau) > 0$ is some small constant specified later; see Lemma 4.19. There exists $\widetilde{h}_i \in C^\infty(A_0, W^{(i)})$ satisfying

$$\|\widetilde{h}_i\|_{\sigma_i,s} \leq \frac{\varepsilon K}{\gamma}\exp(-\widetilde{\chi}^i) \tag{4.57}$$

for some $\tilde\chi \in (1, \bar\chi)$ such that

$$\widetilde{w}_n := \sum_{i=0}^{n}\widetilde{h}_i \in C^\infty(A_0, W^{(n)})$$

satisfies (4.33) and (4.34).

Proof. Let $\varphi: \mathbf{R}\times V_1 \to \mathbf{R}_+$ be a C^∞ function supported in the open ball $B(0, 1)$ of center 0 and radius 1 with $\int_{\mathbf{R}\times V_1}\varphi\;d\mu = 1$. Here μ is the pullback of the Lebesgue measure in $\mathbf{R}^{N+1}$ onto $\mathbf{R}\times V_1$.

Let $\varphi_i: \mathbf{R}\times V_1 \to \mathbf{R}_+$ be the "mollifier"

$$\varphi_i(\lambda) := \Big(\frac{L_i^3}{\nu}\Big)^{N+1}\varphi\Big(\frac{L_i^3}{\nu}\lambda\Big), \qquad \text{where} \qquad \lambda := (\varepsilon, v_1)\,,$$

which is a C^∞ function satisfying

$$\operatorname{supp} \varphi_i \subset B\Big(0, \frac{\nu}{L_i^3}\Big) \qquad \text{and} \qquad \int_{\mathbf{R}\times V_1} \varphi_i \, \mathrm{d}\mu = 1 \,. \tag{4.58}$$

Next we define $\psi_i : \mathbf{R} \times V_1 \to \mathbf{R}_+$ as

$$\psi_i(\lambda) := \Big(\varphi_i * \chi_{A_i^*}\Big)(\lambda) = \int_{\mathbf{R}\times V_1} \varphi_i(\lambda - \eta)\, \chi_{A_i^*}(\eta)\, \mathrm{d}\mu(\eta)\,,$$

where $\chi_{A_i^*}$ is the characteristic function of the set

$$A_i^* := \Big\{\lambda = (\varepsilon, v_1) \in A_i \mid \operatorname{dist}(\lambda, \partial A_i) \geq \frac{\nu}{L_i^3}\Big\} \subset A_i\,,$$

namely $\chi_{A_i^*}(\lambda) := 1$ if $\lambda \in A_i^*$, and $\chi_{A_i^*}(\lambda) := 0$ if $\lambda \notin A_i^*$.

The function ψ_i is C^∞, and $\forall k \in \mathbf{N}$, $\forall \lambda \in \mathbf{R} \times V_1$,

$$\begin{aligned} |D^k \psi_i(\lambda)| &= \Big| \int_{\mathbf{R}\times V_1} D^k \varphi_i(\lambda - \eta)\, \chi_{A_i^*}(\eta)\, \mathrm{d}\mu(\eta) \Big| \\ &\leq \int_{\mathbf{R}\times V_1} \Big| \Big(\frac{L_i^3}{\nu}\Big)^k \Big(\frac{L_i^3}{\nu}\Big)^{N+1} (D^k \varphi)\Big(\frac{L_i^3}{\nu}(\lambda - \eta)\Big) \Big| \, \mathrm{d}\mu(\eta) \\ &= \Big(\frac{L_i^3}{\nu}\Big)^k \int_{\mathbf{R}\times V_1} |D^k \varphi| \, \mathrm{d}\mu = \Big(\frac{L_i^3}{\nu}\Big)^k C(k)\,, \end{aligned} \tag{4.59}$$

where $C(k) := \int_{\mathbf{R}\times V_1} |D^k \varphi| \, \mathrm{d}\mu$. Furthermore, by (4.58) and the definition of A_i^* and $\widetilde{A}_i$,

$$0 \leq \psi_i(\lambda) \leq 1\,, \quad \operatorname{supp} \psi_i \subset \operatorname{int} A_i \quad \text{and} \quad \psi_i(\lambda) = 1 \ \text{ if } \ \lambda \in \widetilde{A}_i\,.$$

Finally, we can define

$$\widetilde{w}_0(\lambda) := w_0(\lambda)\,, \qquad \widetilde{w}_{i+1}(\lambda) := \widetilde{w}_i(\lambda) + \widetilde{h}_{i+1}(\lambda) \in W^{(i+1)}\,,$$

where

$$\widetilde{h}_{i+1}(\lambda) := \begin{cases} \psi_{i+1}(\lambda) h_{i+1}(\lambda) & \text{if } \lambda \in A_{i+1}\,, \\ 0 & \text{if } \lambda \notin A_{i+1}\,, \end{cases}$$

is in $C^\infty(A_0, W^{(i+1)})$ because supp $\psi_{i+1} \subset \operatorname{int} A_{i+1}$ and, by Lemma 4.16, $h_{i+1} \in C^\infty(A_{i+1}, W^{(i+1)})$.

Therefore we have $\widetilde{w}_n(\lambda) := \sum_{i=0}^n \widetilde{h}_i(\lambda)$, $\widetilde{w}_n \in C^\infty(A_0, W^{(n)})$ and (4.33) holds. By the bounds (4.59) and (4.46) we get (4.57) and

$$\Big\| D^k \widetilde{h}_i(\lambda) \Big\|_{\sigma_i, s} \leq C(k, \bar{\chi})^i \Big(\frac{L_{i+1}^3}{\nu}\Big)^k \exp(-\bar{\chi}^i) \leq \frac{K(k)}{\nu^k} \exp(-\widetilde{\chi}^i)$$

for some $1 < \widetilde{\chi} < \bar{\chi}$ and some positive constant $K(k)$ large enough. The estimates (4.34) follow.

The proof of Proposition 4.10 is complete. ■

Proof of Theorem 4.8. The sequence $\widetilde{w}_n$ (with all its derivatives) converges uniformly in A_0 for the norm $\| \ \|_{\bar{\sigma}/2,s}$ to some function

$$\widetilde{w}(\varepsilon, v_1) \in C^\infty(A_0, W \cap X_{\bar{\sigma}/2,s}), \tag{4.60}$$

which, by (4.34), satisfies (4.20) and

$$\left\| \widetilde{w}(\varepsilon, v_1) - \widetilde{w}_n(\varepsilon, v_1) \right\|_{\bar{\sigma}/2,s} \overset{(4.57)}{\leq} \frac{C\varepsilon}{\gamma} \exp(-\widetilde{\chi}^n). \tag{4.61}$$

Note also that by (4.45), we get $\widetilde{w}(t, \cdot) \in H^3(0, \pi)$.

Remark 4.18. If $(\varepsilon, v_1) \notin A_\infty := \cap_{n \geq 0} A_n$, then $\widetilde{w}(\varepsilon, v_1) = \sum_{n \geq 1} \widetilde{h}_n(\varepsilon, v_1)$ is a finite sum; see the proof of Lemma 4.17.

To get the explicit Cantor set B_∞, let us consider

$$\begin{aligned} B_n := \Big\{ (\varepsilon, v_1) \in \widetilde{A}_0 \mid & \left| \omega(\varepsilon) l - j - \varepsilon \frac{M(v_1, \widetilde{w}(\varepsilon, v_1))}{2j} \right| \geq \frac{2\gamma}{(l+j)^\tau}, \\ & |\omega(\varepsilon) l - j| \geq \frac{2\gamma}{(l+j)^\tau}, \ \forall (j, l) \in \mathbf{N} \times \mathbf{N} \text{ such that} \\ & l \neq j, \ \frac{1}{3\varepsilon} < l, \ l \leq L_n, \ j \leq 2L_n \Big\}. \end{aligned}$$

Note that B_n does not depend on the approximate solution w_n but only on the fixed function $\widetilde{w}$.

Lemma 4.19. *If* $\nu\gamma^{-1} > 0$ *and* $\varepsilon\gamma^{-1}$ *are small enough, then*

$$B_n \subset \widetilde{A}_n, \qquad \forall n \geq 0.$$

Proof. We shall prove the lemma by induction. First, $B_0 \subset \widetilde{A}_0$. Suppose next that $B_n \subset \widetilde{A}_n$ holds. In order to prove that $B_{n+1} \subset \widetilde{A}_{n+1}$, take any $(\varepsilon, v_1) \in B_{n+1}$. We have to justify that

$$B\Big((\varepsilon, v_1), \frac{2\nu}{L_{n+1}^3}\Big) \subset A_{n+1}. \tag{4.62}$$

First, since $B_{n+1} \subset B_n \subset \widetilde{A}_n$, then $(\varepsilon, v_1) \in \widetilde{A}_n$. Hence, since $L_{n+1} > L_n$,

$$B\Big((\varepsilon, v_1), \frac{2\nu}{L_{n+1}^3}\Big) \subset A_n.$$

Let $(\varepsilon', v_1') \in B((\varepsilon, v_1), 2\nu/L_{n+1}^3)$. Since $(\varepsilon, v_1) \in \widetilde{A}_n$, we have $\widetilde{w}_n(\varepsilon, v_1) = w_n(\varepsilon, v_1)$. Moreover, by (4.32), $\|Dw_n\|_{\bar{\sigma}/2,s} \leq C$. By (4.61),

$$\begin{aligned} \left\| w_n(\varepsilon', v_1') - \widetilde{w}(\varepsilon, v_1) \right\|_{\bar{\sigma}/2,s} &\leq \left\| w_n(\varepsilon', v_1') - w_n(\varepsilon, v_1) \right\|_{\bar{\sigma}/2,s} \\ &\quad + \left\| w_n(\varepsilon, v_1) - \widetilde{w}(\varepsilon, v_1) \right\|_{\bar{\sigma}/2,s} \\ &\leq \frac{2\nu C}{L_{n+1}^3} + \frac{C\varepsilon}{\gamma} \exp(-\widetilde{\chi}^n). \end{aligned}$$

Hence, by the definition of B_n, setting $\omega' := \sqrt{1-2\varepsilon'}$, we have

$$\begin{aligned}\left|\omega' l - j - \varepsilon' \frac{M(v_1', w_n(\varepsilon', v_1'))}{2j}\right| &\geq \left|\omega l - j - \varepsilon \frac{M(v_1, \widetilde{w}(\varepsilon, v_1))}{2j}\right| - l\frac{C\nu}{L_{n+1}^3} \\ &\quad - C\frac{\varepsilon\nu}{L_{n+1}^3} - C\frac{\varepsilon^2}{\gamma}\exp(-\widetilde{\chi}^n) \\ &\geq \frac{2\gamma}{(l+j)^\tau} - \frac{C\nu}{L_{n+1}^2} - C\frac{\varepsilon^2}{\gamma}\exp(-\widetilde{\chi}^n) \\ &\geq \frac{\gamma}{(l+j)^\tau}\end{aligned}$$

for all $\frac{1}{3\varepsilon} < l < L_{n+1}, l \neq j, j \leq 2L_{n+1}$, whenever

$$\frac{\gamma}{(3L_{n+1})^\tau} \geq C\left(\frac{\nu}{L_{n+1}^2} + \frac{\varepsilon^2}{\gamma}\exp(-\widetilde{\chi}^n)\right). \tag{4.63}$$

We observe that (4.63) holds for $\varepsilon\gamma^{-1}$ and $\nu\gamma^{-1}$ small, for all $n \geq 0$, because $\tau < 2$ and $\lim_{n\to\infty} L_{n+1}^\tau \exp(-\widetilde{\chi}^n) = 0$. Then (4.62) is proved. ■

Corollary 4.20. (Solution of the range equation) *If*

$$(\varepsilon, v_1) \in B_\infty := \bigcap_{n\geq 0} B_n \subset \bigcap_{n\geq 0} \widetilde{A}_n \subset \bigcap_{n\geq 0} A_n\,,$$

then $\widetilde{w}(\varepsilon, v_1) \in X_{\bar{\sigma}/2,s}$ is a solution of the range equation (4.18).

Proof. If $(\varepsilon, v_1) \in B_\infty$ then $(\varepsilon, v_1) \in \cap_{i=0}^n \widetilde{A}_i$, $\forall n \geq 0$, and so $\widetilde{w}_n(\varepsilon, v_1) \overset{(4.33)}{=} w_n(\varepsilon, v_1)$ solves equation (P_n):

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(v_1, w_n) = \varepsilon \Pi_W \Gamma(v_1, w_n) - \varepsilon P_n^\perp \Pi_W \Gamma(v_1, w_n)\,. \tag{4.64}$$

Since

$$\begin{aligned}\left\| P_n^\perp \Pi_W \Gamma(v_1, w_n)\right\|_{\bar{\sigma}/2,s} &\overset{(P2)}{\leq} C\exp\Big(-L_n(\sigma_n - (\bar{\sigma}/2))\Big)\left\|\Gamma(v_1, w_n)\right\|_{\sigma_n,s} \\ &\overset{(P1)}{\leq} C\exp\Big(-\gamma_0\frac{L_0 2^n}{(n^2+1)}\Big)\,,\end{aligned}$$

with $\sigma_n - (\bar{\sigma}/2) \geq \gamma_n := \gamma_0/(n^2+1)$, the right-hand side of (4.64) converges in $X_{\bar{\sigma}/2,s}$ to $\varepsilon\Pi_W\Gamma(v_1, \widetilde{w}(\varepsilon, v_1))$.

Next, using (4.45),

$$\|L_\omega w_n - L_\omega \widetilde{w}\|_{\bar{\sigma}/2,s} \leq \sum_{i>n} \|L_\omega h_i\|_{\bar{\sigma}/2,s}\,,$$

which tends[3] to 0 for $n \to +\infty$. The corollary is proved. ■

[3] The smallness of $\|L_\omega h_i\|_{\bar{\sigma}/2,s}$ can be seen also by $L_\omega h_i = \varepsilon P_{i-1}\Pi_W(\Gamma(v_1, w_i) - \Gamma(v_1, w_{i-1})) + \varepsilon P_{i-1}^\perp P_i \Pi_W \Gamma(v_1, w_i)$.

What remains to prove is that

$$B_\infty := \bigcap_{n \geq 0} B_n \neq \emptyset ,$$

and that in fact, B_∞ is a "large" set.

4.4 Solution of the $(Q1)$-Equation

Once the $(Q2)$-equation and the range equation are solved (with "gaps" for the latter), the last step is to find solutions of the finite-dimensional $(Q1)$-equation

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(\varepsilon, v_1) , \tag{4.65}$$

where

$$\mathcal{G}(\varepsilon, v_1) := f\Big(v_1 + \widetilde{w}(\varepsilon, v_1) + v_2(v_1, \widetilde{w}(\varepsilon, v_1))\Big) ,$$

such that (ε, v_1) belongs to the Cantor set B_∞; see Figure 4.1.

For $\varepsilon = 0$ the $(Q1)$-equation (4.65) reduces to

$$-\Delta v_1 = \Pi_{V_1} \mathcal{G}(0, v_1) = \Pi_{V_1} \Big(a(x)(v_1 + v_2(v_1, 0))^p\Big) . \tag{4.66}$$

Lemma 4.21. $\bar{v}_1 := \Pi_{V_1} \bar{v} \in B(\bar{R}; V_1)$ *is a nondegenerate solution of (4.66).*

Proof. By (4.15) and since $\bar{v}$ solves (4.10), $\bar{v}_1$ solves (4.66). Assume that $h_1 \in V_1$ is a solution of the linearized equation at $\bar{v}_1$ of (4.66), i.e.,

$$-\Delta h_1 = \Pi_{V_1} \Big(pa(x)(\bar{v}_1 + v_2(\bar{v}_1, 0))^{p-1}(h_1 + h_2)\Big) , \tag{4.67}$$

where $h_2 := D_{v_1} v_2(\bar{v}_1, 0) [h_1] \in V_2$. By the definition of v_2,

$$-\Delta v_2(v_1, 0) = \Pi_{V_2} \Big(a(x)(v_1 + v_2(v_1, 0))^p\Big) , \qquad \forall v_1 \in B(2\bar{R}, V_1) ,$$

whence, differentiating at $\bar{v}_1$,

$$-\Delta h_2 = \Pi_{V_2} \Big(pa(x)(\bar{v}_1 + v_2(\bar{v}_1, 0))^{p-1}(h_1 + h_2)\Big) . \tag{4.68}$$

Summing (4.67) and (4.68), $h = h_1 + h_2 \in V$ is a solution of $\Phi''(\bar{v}) [h] = 0$. By (4.12), $h = 0$ and hence $h_1 = 0$. ∎

Lemma 4.22. *There exists a* C^∞ *path*

$$\varepsilon \mapsto v_1(\varepsilon) \in B(2\bar{R}, V_1)$$

of solutions of the equation (4.65) with $v_1(0) = \bar{v}_1$.

Proof. We may apply the implicit function theorem by Lemma 4.21 and since

$$(\varepsilon, v_1) \mapsto -\Delta v_1 - \Pi_{V_1} \mathcal{G}(\varepsilon, v_1)$$

is in $C^\infty([0, \varepsilon_0) \times B(2\bar{R}, V_1); V_1)$. ■

By Theorem 4.8 and Proposition 4.5,

$$u(\varepsilon) := \varepsilon^{\frac{1}{p-1}} \Big[v_1(\varepsilon) + v_2\Big(v_1(\varepsilon), \widetilde{w}(\varepsilon, v_1(\varepsilon))\Big) + \widetilde{w}(\varepsilon, v_1(\varepsilon)) \Big] \in X_{\bar{\sigma}/2,s} \tag{4.69}$$

is a solution of equation (4.1) *if* ε belongs to the Cantor-like set

$$\mathcal{C} := \Big\{ \varepsilon \in [0, \varepsilon_0) \mid (\varepsilon, v_1(\varepsilon)) \in B_\infty \Big\} .$$

Remark 4.23. Actually, u is a classical solution because

$$u_{xx}(t, x) = \omega^2 u_{tt}(t, x) - f(x, u(t, x)) \in H_0^1(0, \pi)$$

$\forall t \in \mathbf{T}$, and so $u(t, \cdot) \in H^3(0, \pi) \subset C^2(0, \pi)$.

The smoothness of $v_1(\cdot)$ actually implies that the set $\mathcal{C}$ has full density at the origin, namely

$$\lim_{\eta \to 0^+} \frac{|\mathcal{C} \cap [0, \eta)\,|}{\eta} = 1 . \tag{4.70}$$

Geometrically this estimate exploits the structure of the Cantor set B_∞ and that the curve of solutions $\varepsilon \mapsto v_1(\varepsilon)$ crosses B_∞ transversally (it is a graph); see Figure 4.1.

Proposition 4.24. *The Cantor set $\mathcal{C}$ satisfies the measure estimate (4.70).*

Proof. Let $0 < \eta < \varepsilon_0$. We have to estimate the complementary set

$$\mathcal{C}^c \cap (0, \eta) = \Big\{ \varepsilon \in (0, \eta) \Big| \ \Big| \omega(\varepsilon) l - j - \frac{\varepsilon m(\varepsilon)}{2j} \Big| < \frac{2\gamma}{(l + j)^\tau} \\ \text{or } \Big| \omega(\varepsilon) l - j \Big| < \frac{2\gamma}{(l + j)^\tau} \text{ for some } l \geq \frac{1}{3\varepsilon} ,\ l \neq j \Big\} ,$$

where

$$m(\varepsilon) := M(v_1(\varepsilon), \widetilde{w}(\varepsilon, v_1(\varepsilon)))$$

is a function in $C^\infty([0, \varepsilon_0)\, , \mathbf{R})$ because $\varepsilon \mapsto v_1(\varepsilon)$ is C^∞ by Lemma 4.22, $\widetilde{w}(\varepsilon, v_1)$ is C^∞ by Theorem 4.8, and $M(\cdot, \cdot)$, defined in (4.21), is C^∞. Actually, in the next argument we just use that $\varepsilon \mapsto m(\varepsilon)$ is C^1.

We can write

$$\mathcal{C}^c \cap (0, \eta) \subset \bigcup_{(l,j) \in \mathcal{I}_R} \Big(\mathcal{R}_{l,j} \cup \mathcal{S}_{l,j} \Big) , \tag{4.71}$$

where

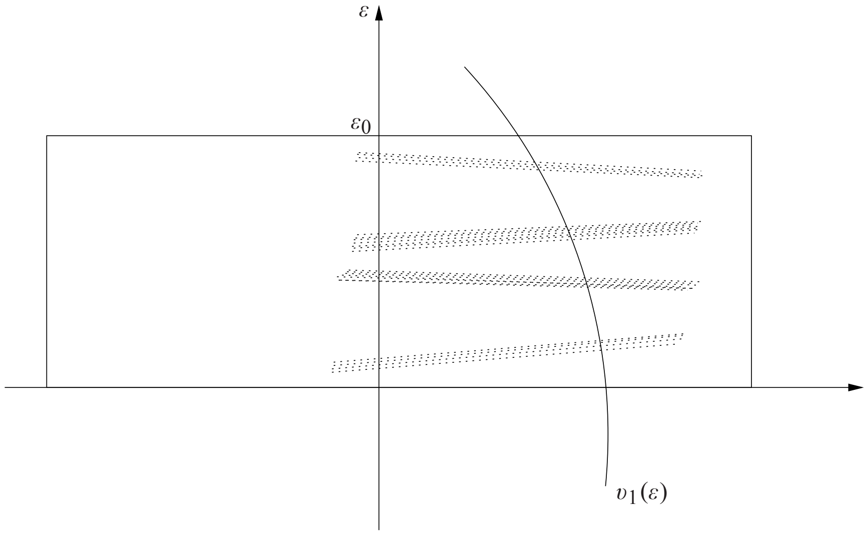

Fig. 4.1. The Cantor set in which the range equation is solved and the solutions $v_1(\varepsilon)$ of the bifurcation equation.

$$\mathcal{S}_{l,j} := \left\{ \varepsilon \in (0, \eta) \mid \left| \omega(\varepsilon) l - j - \frac{\varepsilon m(\varepsilon)}{2j} \right| < \frac{2\gamma}{(l+j)^\tau} \right\},$$
$$\mathcal{R}_{l,j} := \left\{ \varepsilon \in (0, \eta) \mid \left| \omega(\varepsilon) l - j \right| < \frac{2\gamma}{(l+j)^\tau} \right\},$$

and

$$\mathcal{I}_R := \left\{ (l, j) \mid l > \frac{1}{3\eta},\ l \neq j,\ \frac{j}{l} \in [1 - 4\eta, 1 + 4\eta] \right\}$$

(note indeed that $\mathcal{R}_{j,l} = \mathcal{S}_{j,l} = \emptyset$ unless $j/l \in [1 - 4\eta, 1 + 4\eta]$).

By Proposition 2.27 we already know that

$$\Big| \bigcup_{(l,j) \in \mathcal{I}_R} \mathcal{R}_{l,j} \Big| = o(\eta)$$

for $\eta \to 0$. Here we show that also,

$$\Big| \bigcup_{(l,j) \in \mathcal{I}_R} \mathcal{S}_{l,j} \Big| = o(\eta)\,. \tag{4.72}$$

This is a consequence of the measure estimate

$$|\mathcal{S}_{lj}| = O\Big(\frac{\gamma}{l^{\tau+1}} \Big)\,, \tag{4.73}$$

which follows by the C^1-smoothness of $m(\varepsilon)$. Indeed, setting

$$f_{lj}(\varepsilon) := \omega(\varepsilon)l - j - \frac{\varepsilon m(\varepsilon)}{2j},$$

we have, recalling $\omega(\varepsilon) = \sqrt{1-2\varepsilon}$,

$$|\partial_\varepsilon f_{lj}(\varepsilon)| = \left|\frac{-l}{\sqrt{1-2\varepsilon}} - \frac{m(\varepsilon)+\varepsilon m'(\varepsilon)}{2j}\right| \geq \frac{l}{2}$$

for any $\varepsilon \in (0,\eta)$ and η small enough.

Since for a given l, the number of j for which $(l,j) \in \mathcal{I}_R$ is $O(\eta l)$, we have

$$\Big|\bigcup_{(l,j)\in\mathcal{I}_R} \mathcal{S}_{l,j}\Big| \leq \sum_{(l,j)\in\mathcal{I}_R} |\mathcal{S}_{l,j}| \leq C \sum_{l\geq 1/3\eta} \eta l \times \frac{\gamma}{l^{\tau+1}} \leq C\gamma\eta^\tau,$$

which concludes the proof of (4.72) because $\tau > 1$. ■

Remark 4.25. In [27] is proved an asymptotically full measure intersection property as in Proposition 4.24 under just a weak BV dependence of $\varepsilon \mapsto v_1(\varepsilon)$. This is relevant for weakening the nondegeneracy condition (4.12) as discussed in Remark (4.28) below.

The nondegeneracy condition (4.12) on the nonlinearity can be verified on several examples,[4] yielding the following theorem.

Theorem 4.26. ([26]) *Let*

$$f(x,u) = \begin{cases} a_2u^2, & a_2 \neq 0, \\ a_3(x)u^3, & \langle a_3\rangle := (1/\pi)\int_0^\pi a_3(x) \neq 0, \\ a_4u^4, & a_4 \neq 0. \end{cases}$$

Then, $s > 1/2$ being given, there exist $\varepsilon_0 > 0$, $\bar\sigma > 0$, and a C^∞-curve $[0,\varepsilon_0) \ni \varepsilon \mapsto u(\varepsilon) \in X_{\bar\sigma/2,s}$ with the following properties:

(*i*) $\left\|u(\varepsilon) - \varepsilon^{\frac{1}{p-1}}\bar v\right\|_{\bar\sigma/2,s} = O(\varepsilon^{\frac{2}{p-1}})$ *for some* $\bar v \in V \cap X_{\bar\sigma,s}$, $\bar v \neq \{0\}$.

(*ii*) *There exists a Cantor set $\mathcal{C} \subset [0,\varepsilon_0)$ of asymptotically full measure, i.e., satisfying (4.70), such that $\forall\, \varepsilon \in \mathcal{C}$, $u(\varepsilon)(\omega(\varepsilon)t,x)$ is a $2\pi/\omega(\varepsilon)$-periodic classical solution of (4.1) with frequency respectively*

$$\omega = \begin{cases} \sqrt{1-2\varepsilon^2}, \\ \sqrt{1-2\varepsilon\,\mathrm{sign}\langle a_3\rangle}, \\ \sqrt{1-2\varepsilon^2}. \end{cases}$$

[4] In the cases $f = a_2u^2$, a_4u^4 the condition (4.7) is violated, and as in Section 2.4, the correct 0th-order bifurcation equation is the Euler–Lagrange equation of $\frac{1}{2}\|v\|^2_{H_1} - a_p^2\int_\Omega v^p\Box^{-1}v^p$, $p = 2,4$.

Remark 4.27. **(Multiplicity)** To get multiplicity of periodic solutions we look for $2\pi/n$ time-periodic solutions of (4.2). We can prove ([26], [11]) that there exist $n_0 \in \mathbf{N}$ and a Cantor-like set $\mathcal{C}$ of asymptotically full measure such that $\forall \varepsilon \in \mathcal{C}$, equation (4.1) possesses $2\pi/(n\omega(\varepsilon))$-periodic, geometrically distinct solutions u_n for every

$$n_0 \leq n \leq N(\varepsilon) \quad \text{where} \quad \lim_{\varepsilon \to 0} N(\varepsilon) = +\infty .$$

Each u_n is in particular $2\pi/\omega(\varepsilon)$-periodic.

Remark 4.28. **(Weaken the nondegeneracy condition)** We could solve the bifurcation equation with variational methods as in Chapter 2, proving, $\forall \varepsilon > 0$ small, the existence of mountain pass solutions $v_1(\varepsilon)$ of the $(Q1)$-equation. However, since the section

$$E_\varepsilon := \Big\{ v_1 \mid (\varepsilon, v_1) \in B_\infty \Big\}$$

has "gaps" (except for ω in the zero-measure set $\mathcal{W}_\gamma$ of Chapter 2), the big difficulty is to prove that $(\varepsilon, v_1(\varepsilon)) \in B_\infty$ for a large set of ε's.

Although B_∞ is in some sense a "large" set, this property is not obvious: the complement of B_∞ is actually arcwise connected! And the critical point $v_1(\varepsilon)$ could depend (without a nondegeneracy condition) in a highly irregular way on ε.

Results in this direction have been obtained in [27] using variational methods for parameter-dependent families of functionals possessing the mountain pass geometry.

4.5 The Linearized Operator

We prove in this section the key property (P3) on the inversion of the linear operator $\mathcal{L}_n(\varepsilon, v_1, w)$ defined in (4.24).

4.5.1 Decomposition of $\mathcal{L}_n$

We can write, recalling (4.19),

$$\begin{aligned} &\mathcal{L}_n(\varepsilon, v_1, w)[h] \\ &\quad := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(v_1, w)[h] \\ &\quad = L_\omega h - \varepsilon P_n \Pi_W ((\partial_u f)(v_1 + w + v_2(v_1, w))(h + \partial_w v_2[h])) \\ &\quad = L_\omega h - \varepsilon P_n \Pi_W (b(t,x)h) - \varepsilon P_n \Pi_W \Big(b(t,x) \partial_w v_2[h] \Big), \end{aligned} \tag{4.74}$$

where for brevity,

$$b(t,x) := (\partial_u f)(x, v_1 + w + v_2(v_1, w)) . \tag{4.75}$$

To invert $\mathcal{L}_n$ it is convenient to perform a Fourier expansion and represent the operator $\mathcal{L}_n$ as a matrix. The main difference with respect to the procedure of Craig–Wayne–Bourgain [51, 36] is that we shall develop $\mathcal{L}_n$ only in the *time*-Fourier basis and not in the time and spatial Fourier basis $\{e^{ilt}\sin(jx),\ l \in \mathbf{Z},\ j \geq 1\}$. This is more convenient in dealing with nonlinearities $f(x,u)$ with finite regularity in x and without oddness assumptions.

In the time-Fourier basis the operator[5]

$$L_\omega := -\omega^2\partial_{tt} + \partial_{xx}$$

is represented by the diagonal matrix (of spatial operators)

$$L_\omega \equiv \begin{pmatrix} \omega^2 L_n^2 + \partial_{xx} & 0 & \dots & 0 & 0 \\ \dots & \dots & & & \dots \\ 0 & 0 & \omega^2 k^2 + \partial_{xx} & & 0 \\ \dots & & & \dots & \\ 0 & 0 & 0 & & \omega^2 L_n^2 + \partial_{xx} \end{pmatrix}$$

because

$$h = \sum_{|k|\leq L_n} \exp(ikt) h_k(x) \quad \Rightarrow \quad L_\omega h = \sum_{|k|\leq L_n} \exp(ikt)(\omega^2 k^2 + \partial_{xx}) h_k(x)\,.$$

The operator $h \mapsto P_n \Pi_W (b(t,x)\,h)$ is the composition of the multiplication operator for the function

$$b(t,x) = \sum_{l\in\mathbf{Z}} \exp(ilt) b_l(x)$$

with the projectors Π_W and P_n.

The projector $\Pi_W \colon X_{\sigma,s} \mapsto W$ is given, $\forall\, h = \sum_{k\in\mathbf{Z}} \exp(ikt) h_k \in X_{\sigma,s}$, by

$$(\Pi_W h)(t,x) = \sum_{k\in\mathbf{Z}} \exp(ikt)(\pi_k h_k)(x)\,, \tag{4.76}$$

where

$$\pi_k \colon H_0^1((0,\pi);\mathbf{R}) \mapsto \langle \sin kx \rangle^\perp$$

is the L^2-orthogonal projector.

For all $h = \sum_{|k|\leq L_n} \exp(ikt) h_k$ we have

$$P_n \Pi_W (b(t,x)\,h) = \sum_{|k|,|l|\leq L_n} \exp(ilt)\pi_l (b_{l-k}(x) h_k(x))\,,$$

so that it is represented by the matrix that has in row l and column k the space operator $\pi_l(b_{l-k}(x)\,[\cdot\,])$, namely

$$P_n \Pi_W (b\,\cdot) \equiv \begin{pmatrix} \pi_{-L_n} b_0 & \pi_{-L_n} b_{-1} & \dots & & & \pi_{-L_n} b_{-2L_n} \\ \dots & & & & & \\ & & \pi_l b_0 & \dots & \pi_l b_{l-k} & \\ \dots & & & \dots & & \\ \dots & & & \dots & & \dots \\ \pi_{L_n} b_{2L_n} & & & \dots & \pi_{L_n} b_1 & \pi_{L_n} b_0 \end{pmatrix}.$$

[5] In order to have the positive signs below it is the opposite of L_ω defined in (4.6). Since there is no risk of confusion we denote it in the same way.

As usual, in the Fourier basis, the multiplication operator is described by a "Toepliz matrix" (constant terms along the parallels to the diagonal). Note also that the zero time Fourier coefficient

$$b_0(x) = \frac{1}{2\pi} \int_0^{2\pi} b(t,x)\,\mathrm{d}t$$

is the time average of $b(t,x)$.

Distinguishing the "diagonal" matrix D,

$$D \equiv \begin{pmatrix} \omega^2 L_n^2 + \partial_{xx} - \varepsilon\pi_{-L_n} b_0 & 0 & \dots & 0 & 0 \\ \dots & \dots & & & \dots \\ 0 & 0 & \omega^2 k^2 + \partial_{xx} - \varepsilon\pi_k b_0 & & 0 \\ \dots & & & \dots & \\ 0 & 0 & 0 & & \omega^2 L_n^2 + \partial_{xx} - \varepsilon\pi_{L_n} b_0 \end{pmatrix},$$

and the "off-diagonal Toepliz" matrix $\mathcal{M}_1$,

$$\mathcal{M}_1 \equiv \begin{pmatrix} 0 & \pi_{-L_n} b_{-1} \dots & & \pi_{-L_n} b_{-2L_n} \\ \dots & & & \\ & 0 & \dots\ \pi_l b_{l-k} & \\ & & & \\ \dots & & & \\ \pi_{L_n} b_{2L_n} & & \dots\ \pi_{L_n} b_1 & 0 \end{pmatrix},$$

we decompose

$$\mathcal{L}_n(\varepsilon, v_1, w) = D - \varepsilon\mathcal{M}_1 - \varepsilon\mathcal{M}_2\,,$$

where

$$\begin{cases} Dh := L_\omega h - \varepsilon P_n \Pi_W (b_0(x)\,h)\,, \\ \mathcal{M}_1 h := P_n \Pi_W (\bar{b}(t,x)\,h)\,, \\ \mathcal{M}_2 h := P_n \Pi_W (b(t,x)\,\partial_w v_2\,[h])\,, \end{cases} \tag{4.77}$$

and

$$\bar{b}(t,x) := b(t,x) - b_0(x)$$

has zero time-average. Note that we do not decompose the term $\mathcal{M}_2$ into diagonal and "off-diagonal terms."

To invert $\mathcal{L}_n$ we first (Step 1) prove that the diagonal part D is invertible; see Corollary 4.33. Next (Step 2) we prove that the "off-diagonal Toepliz" operators $\varepsilon\mathcal{M}_1$ (Lemma 4.37) and $\varepsilon\mathcal{M}_2$ (Lemma 4.38) are small enough with respect to D, yielding the invertibility of the whole of $\mathcal{L}_n$.

4.5.2 Step 1: Inversion of D

In the time-Fourier basis the operator D is represented by the diagonal matrix

$$D = \begin{pmatrix} D_{-L_n} & 0 & \dots & 0 & 0 \\ \dots & \dots & & & 0 \\ 0 & 0 & D_k & & 0 \\ \dots & & & \dots & \\ 0 & 0 & 0 & & D_{L_n} \end{pmatrix},$$

where each

$$D_k : \mathcal{D}(D_k) \subset \langle \sin kx \rangle^{\perp} \mapsto \langle \sin kx \rangle^{\perp}$$

is the Sturm–Liouville-type operator

$$D_k u := \omega^2 k^2 u + \partial_{xx} u - \varepsilon \pi_k (b_0(x)\, u)\,.$$

We can diagonalize in space each D_k with respect to the scalar product (for $\varepsilon |b_0|_\infty < 1$)

$$\langle u, v \rangle_\varepsilon := \int_0^\pi u_x v_x + \varepsilon b_0(x) u v \, \mathrm{d}x\,.$$

Its associated norm is equivalent to the H^1-norm

$$\|u\|_{H^1}^2 \Big(1 - \varepsilon\, |b_0|_\infty\Big) \le \|u\|_\varepsilon^2 \le \|u\|_{H^1}^2 \Big(1 + \varepsilon\, |b_0|_\infty\Big) \tag{4.78}$$

because $\int_0^\pi u^2 \le \int_0^\pi u_x^2$, $\forall u \in H_0^1(0, \pi)$.

Lemma 4.29. (Sturm–Liouville) *The operator $-\partial_{xx} + \varepsilon \pi_k (b_0(x)\ \cdot)$ acting on $\langle \sin kx \rangle^{\perp}$ possesses a $\langle\ ,\ \rangle_\varepsilon$-orthonormal basis of eigenvectors $(\varphi_{k,j})_{j \ge 1, j \ne |k|}$ with eigenvalues*

$$\lambda_{k,j} = \lambda_{k,j}(\varepsilon, v_1, w) = j^2 + \varepsilon M(v_1, w) + O\Big(\frac{\varepsilon \|b_0\|_{H^1}}{j}\Big), \quad j \ne |k|\,, \tag{4.79}$$

$\lambda_{k,j} = \lambda_{-k,j}$, $\varphi_{-k,j} = \varphi_{k,j}$, where

$$M(v_1, w) = \frac{1}{\pi} \int_0^\pi b_0(x)\, \mathrm{d}x = \frac{1}{2\pi^2} \int_\Omega b(t, x)\, \mathrm{d}x \mathrm{d}t$$

was defined in (4.21).

Proof. It is a standard fact of perturbation theory for the eigenvalues of self-adjoint operators; see, e.g., [77]. The derivative with respect to ε at $\varepsilon = 0$ of an eigenvalue $\lambda_{kj}(\varepsilon)$ is obtained restricting the derivative of the operators with respect to ε, here $\pi_k(b_0(x)\,\cdot)$, to the unperturbed eigenspace, here $\sin(jx)$,

$$\begin{aligned} \partial_\varepsilon \lambda_{kj}(\varepsilon)|_{\varepsilon=0} &= \frac{2}{\pi} \int_0^\pi \sin(jx) \pi_k (b_0(x)\ \sin(jx)) = \frac{2}{\pi} \int_0^\pi \sin^2(jx) b_0(x) \\ &= \frac{1}{\pi} \int_0^\pi (1 - \cos(2jx)) b_0(x) = \frac{1}{\pi} \int_0^\pi b_0(x) + O\Big(\frac{\|b_0\|_{H^1}}{j}\Big)\,, \end{aligned}$$

and integrating by parts. For details see Lemma 4.1 in [26]. ■

Corollary 4.30. (Diagonalization of D**)** *The operator* D *is the diagonal operator* $\mathrm{diag}\{\omega^2 k^2 - \lambda_{k,j}\}$ *in the basis*

$$\{\cos(kt)\varphi_{k,j}\ ; 0 \le k \le L_n\ , j \ge 1, j \ne k\}\,.$$

Imposing the "nonresonance conditions" of property (P3), we get a lower bound for the modulus of the eigenvalues of D_k.

Lemma 4.31. (Lower bound for the eigenvalues of D**)** *There is* $c > 0$ *such that if* $(\varepsilon, v_1) \in \mathcal{A}_n(w)$ *and* ε_0 *is small enough, then* $\forall\ 1 \le |k| \le L_n$,

$$\alpha_k := \min_{j \ge 1, j \ne |k|} |\omega^2 k^2 - \lambda_{k,j}| \ge \frac{c\,\gamma}{|k|^{\tau-1}} > 0\,. \tag{4.80}$$

Moreover, $\alpha_0 \ge 1/2$.

Proof. Since $\alpha_{-k} = \alpha_k$, it is sufficient to consider $k \ge 0$. The estimate for α_0 is trivial. Next we have two cases to consider.

First Case: $0 < k \le \frac{1}{3\varepsilon}$. Since $|\omega - 1| \le 2\varepsilon$ and $k \ne j$, we have

$$|\omega k - j| \ge |k - j| - |\omega - 1| k \ge 1 - 2\varepsilon \frac{1}{3\varepsilon} = \frac{1}{3}\,.$$

Therefore $|\omega^2 k^2 - j^2| = |\omega k - j|(\omega k + j) \ge 1/3$, and by (4.79), for ε small,

$$|\omega^2 k^2 - \lambda_{kj}| \ge \frac{1}{3} \ge \frac{c\,\gamma}{k^{\tau-1}}$$

(the singular sites appear only for $k > \frac{1}{3\varepsilon}$).

Second Case: $k > \frac{1}{3\varepsilon}$. By a Taylor expansion,

$$\begin{aligned}
|\omega^2 k^2 - \lambda_{k,j}| &\overset{(4.79)}{=} \Big|\omega^2 k^2 - j^2 - \varepsilon M(v_1, w) + O\Big(\frac{\varepsilon \|b_0\|_{H^1}}{j}\Big)\Big| \\
&\ge \Big|\Big(\omega k - \sqrt{j^2 + \varepsilon M(v_1, w)}\Big)\Big(\omega k + \sqrt{j^2 + \varepsilon M(v_1, w)}\Big)\Big| \\
&\quad - \frac{C\varepsilon}{j} \\
&\ge \Big|\omega k - j - \varepsilon \frac{M(v_1, w)}{2j} + O\Big(\frac{\varepsilon^2}{j^3}\Big)\Big| \omega k - C\frac{\varepsilon}{j} \\
&\ge \Big|\omega k - j - \varepsilon \frac{M(v_1, w)}{2j}\Big|\ \omega k - C'\Big(\frac{\varepsilon^2 k}{j^3} + \frac{\varepsilon}{j}\Big) \\
&\ge \frac{\gamma\,\omega k}{(k+j)^\tau} - C\Big(\frac{\varepsilon^2 k}{j^3} + \frac{\varepsilon}{j}\Big),
\end{aligned} \tag{4.81}$$

since $(\varepsilon, v_1) \in \mathcal{A}_n(w)$.

If $\alpha_k := \min_{j \ge 1, j \ne k} |\omega^2 k^2 - \lambda_{k,j}|$ is attained at $j = j(k)$ then $|\omega k - j| \le 1$ (provided ε is small enough). Therefore, using $1 < \tau < 2$ and $|\omega - 1| \le 2\varepsilon$, we can derive (4.80) from (4.81). ∎

Lemma 4.32. *Suppose* $\alpha_k \neq 0$. *Then* D_k *is invertible and* $\forall u = \sum_{j \neq |k|} u_j \varphi_{k,j} \in \langle \sin kx \rangle^{\perp}$,

$$|D_k|^{-1/2} u := \sum_{j \neq |k|} \frac{u_j \, \varphi_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}}$$

satisfies

$$\left\| |D_k|^{-1/2} u \right\|_{H^1} \leq \frac{2}{\sqrt{\alpha_k}} \|u\|_{H^1} . \tag{4.82}$$

Proof. Using that $\varphi_{k,j}$ is an orthonormal basis for the $\langle \, , \, \rangle_\varepsilon$ scalar product yields

$$\begin{aligned} \left\| \, |D_k|^{-1/2} u \, \right\|_\varepsilon^2 &= \left\| \sum_{j \neq |k|} \frac{u_j \, \varphi_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}} \right\|_\varepsilon^2 = \sum_{j \neq |k|} \frac{|u_j|^2}{|\omega^2 k^2 - \lambda_{k,j}|} \\ &\leq \frac{1}{\alpha_k} \sum_{j \neq |k|} |u_j|^2 = \frac{\|u\|_\varepsilon^2}{\alpha_k} . \end{aligned}$$

Since the norms $\| \cdot \|_\varepsilon$ and $\| \cdot \|_{H^1}$ are equivalent, (4.82) follows. ∎

Corollary 4.33. (Estimate of $|D|^{-1/2}$) *If* $(\varepsilon, v_1) \in \mathcal{A}_n(w)$ *and* ε_0 *is small enough, then* D *is invertible and* $\forall s' \geq 0$,

$$\left\| |D|^{-1/2} h \right\|_{\sigma,s'} \leq \frac{C}{\sqrt{\gamma}} \|h\|_{\sigma, s' + \frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}. \tag{4.83}$$

Proof. Since

$$|D|^{-1/2} h := \sum_{|k| \leq L_n} \exp(\mathrm{i}kt) |D_k|^{-1/2} h_k ,$$

using (4.82) and (4.80),

$$\begin{aligned} \left\| |D|^{-1/2} h \right\|_{\sigma,s'}^2 &= \sum_{|k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \left\| |D_k|^{-1/2} h_k \right\|_{H^1}^2 \\ &\leq \sum_{|k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{4}{\alpha_k} \|h_k\|_{H^1}^2 \\ &\leq 8 \|h_0\|_{H^1}^2 + C \sum_{0 < |k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{|k|^{\tau-1}}{\gamma} \|h_k\|_{H^1}^2 \\ &\leq \frac{C'}{\gamma} \|h\|_{\sigma, s' + \frac{\tau-1}{2}}^2 , \end{aligned}$$

proving (4.83). ∎

4.5.3 Step 2: Inversion of $\mathcal{L}_n$

To show the invertibility of $\mathcal{L}_n$ it is convenient to write

$$\mathcal{L}_n = D - \varepsilon \mathcal{M}_1 - \varepsilon \mathcal{M}_2 = |D|^{1/2}\Big(U - \varepsilon \mathcal{R}_1 - \varepsilon \mathcal{R}_2\Big)|D|^{1/2},$$

where

$$U := |D|^{-1/2} D |D|^{-1/2} = |D|^{-1} D$$

and

$$\mathcal{R}_i := |D|^{-1/2} \mathcal{M}_i |D|^{-1/2}, \quad i = 1, 2.$$

Lemma 4.34. (Estimate of $\|U^{-1}\|$) *U is invertible and $\forall s' \geq 0$,*

$$\Big\|U^{-1} h\Big\|_{\sigma,s'} = \|h\|_{\sigma,s'}\Big(1 + O(\varepsilon \|b_0\|_{H^1})\Big) \qquad \forall\, h \in W^{(n)}. \tag{4.84}$$

Proof. The eigenvalues of $U_k := |D_k|^{-1} D_k$ are $\mathrm{sign}(\omega^2 l^2 - \lambda_{lj}) = \pm 1$ and therefore U_k is invertible and

$$\|U_k^{-1} u\|_\varepsilon = \|u\|_\varepsilon, \quad \forall u \in \langle \sin kx \rangle^\perp.$$

By (4.78),

$$\|U_k^{-1} u\|_{H^1} = \|u\|_\varepsilon (1 + O(\varepsilon \|b_0\|_{H^1})).$$

Therefore U is invertible and (4.84) holds. ∎

The estimate of the “off-diagonal” operator $\mathcal{R}_1$ requires a careful analysis of the “*small divisors*.”

Lemma 4.35. (Analysis of the small divisors) *Let $(\varepsilon, v_1) \in \mathcal{A}_n(w)$ and ε_0 small. There exists $C > 0$ such that $\forall l \neq k$,*

$$\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2 \varepsilon^{\tau-1}}, \qquad \textit{where} \qquad \beta := \frac{2-\tau}{\tau} \in (0,1). \tag{4.85}$$

Proof. To obtain (4.85) we distinguish different cases.

- FIRST CASE: *sites far away from the diagonal:* $|k-l| \geq (1/2)\,[\max(|k|,|l|)]^\beta$. Then $(\alpha_k \alpha_l)^{-1} \leq C|k-l|^{2\frac{\tau-1}{\beta}}/\gamma^2$.

Estimating both α_k, α_l with the worst possible lower bound (4.80),

$$\alpha_k \geq \frac{c}{\gamma\, |k|^{\tau-1}}, \; \alpha_l \geq \frac{c}{\gamma\, |l|^{\tau-1}},$$

we obtain

$$\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k|^{\tau-1} |l|^{\tau-1}}{\gamma^2} \leq C \frac{[\max(|k|,|l|)]^{2(\tau-1)}}{\gamma^2} \leq C' \frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2}.$$

The other cases to consider (sites close to the diagonal),

$$0 < |k-l| < \frac{1}{2}\Big[\max(|k|,|l|)\Big]^{\beta}, \tag{4.86}$$

are the most dangerous.

We note that in this situation, $\operatorname{sign}(l) = \operatorname{sign}(k)$, and to fix the ideas, we assume in the sequel that $l, k \geq 0$ (the estimate for $k, l < 0$ is the same, since $\alpha_k \alpha_l = \alpha_{-k}\alpha_{-l}$). In this case, $k, l \geq 0$ are of the same order, namely

$$\frac{l}{2} \leq k \leq 2l \qquad \text{and} \qquad \frac{k}{2} \leq l \leq 2k\,.$$

Indeed, if $k > l$, then $k - l < k^{\beta}/2 \leq k/2$, since $\beta < 1$, and therefore $k \leq 2l$. If $k < l$, then $l - k < l^{\beta}/2 \leq l/2$, whence $k \geq l/2$.

- SECOND CASE: $0 < |k-l| < (1/2)\,[\max(|k|,|l|)]^{\beta}$ and ($|k| \leq \frac{1}{3\varepsilon}$ or $|l| \leq \frac{1}{3\varepsilon}$). Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma$.

Suppose, for example, that $0 \leq k \leq \frac{1}{3\varepsilon}$. We claim that if ε is small enough, then $\alpha_k \geq (k+1)/8$. Indeed, $\forall j \neq k$,

$$|\omega k - j| = |\omega k - k + k - j| \geq |k - j| - |\omega - 1|\,|k| \geq 1 - 2\varepsilon\, k \geq \frac{1}{3}.$$

Therefore $\forall 0 \leq k \leq \frac{1}{3\varepsilon}$, $\forall j \neq k$, $j \geq 1$,

$$|\omega^2 k^2 - j^2| = |\omega k - j|\,|\omega k + j| \geq \frac{\omega k + 1}{3} \geq \frac{k+1}{6},$$

and so

$$\begin{aligned} \alpha_k &:= \min_{j\geq 1, k\neq j} \Big|\omega^2 k^2 - \lambda_{k,j}\Big| \\ &= \min_{j\geq 1, k\neq j} \Big|\omega^2 k^2 - j^2 - \varepsilon M(v_1, w) + O\Big(\frac{\varepsilon\|b_0\|_{H^1}}{j}\Big)\Big| \\ &\geq \frac{k+1}{6} - \varepsilon\, C \geq \frac{k+1}{8}\,. \end{aligned} \tag{4.87}$$

Next, we estimate α_l. If also $0 \leq l \leq \frac{1}{3\varepsilon}$, then arguing as above, $\alpha_l \geq (l+1)/8 \geq 1/8$ and therefore $(\alpha_k\alpha_l)^{-1} \leq 64$.

If $l > \frac{1}{3\varepsilon}$, we estimate $\alpha_l \geq c\gamma/l^{\tau-1}$ by the lower bound (4.80), and (4.87) implies, since $l \leq 2k$,

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{l^{\tau-1}}{(k+1)\gamma} \leq \frac{C'}{\gamma}\,\frac{k^{\tau-1}}{(k+1)} \leq \frac{C'}{\gamma}$$

because $1 < \tau < 2$.

In the remaining cases we consider

$$|k-l| < \frac{1}{2}\Big[\max(|k|,|l|)\Big]^{\beta} \quad \text{and both } |k|,|l| > \frac{1}{3\varepsilon}\,.$$

We have to distinguish two subcases. For this, $\forall k \in \mathbf{Z}$, let $j = j(k) \geq 1$ be an integer such that $\alpha_k := \min_{n \neq |k|} |\omega^2 k^2 - \lambda_{k,n}| = |\omega^2 k^2 - \lambda_{k,j}|$. Analogously let $i = i(k) \geq 1$ be an integer such that $\alpha_l = |\omega^2 l^2 - \lambda_{l,i}|$.

- THIRD CASE: $0 < |k-l| < (1/2)\,[\max(|k|,|l|)]^{\beta}$, $|k|,|l| > \frac{1}{3\varepsilon}$ and $k-l = j-i$. Then $(\alpha_k \alpha_l)^{-1} \leq C/\gamma\,\varepsilon^{\tau-1}$.

  Indeed,

$$\Big|(\omega k - j) - (\omega l - i)\Big| = |\omega(k-l) - (j-i)| = |\omega - 1||k-l| \geq \frac{\varepsilon}{2}\,,$$

and therefore $|\omega k - j| \geq \varepsilon/4$ or $|\omega l - i| \geq \varepsilon/4$. Assume for instance that $|\omega k - j| \geq \varepsilon/4$. Then

$$|\omega^2 k^2 - j^2| = |\omega k - j|\,|\omega k + j| \geq \varepsilon \frac{\omega k}{4} \geq \varepsilon(1-2\varepsilon)\frac{k}{4}\,,$$

and so for ε small enough, $|\alpha_k| \geq \varepsilon k/8$. Hence, estimating $\alpha_l \geq c\gamma/l^{\tau-1}$ with the worst possible lower bound (4.80) yields

$$\frac{1}{\alpha_k \alpha_l} \leq C\frac{l^{\tau-1}}{\gamma\,\varepsilon k} \leq \frac{C'}{\gamma\,k^{2-\tau}\varepsilon} \leq \frac{C'}{\gamma\,\varepsilon^{\tau-1}}$$

because $l \leq 2k$ and $k > \frac{1}{3\varepsilon}$.

- FOURTH CASE: $0 < |k-l| < (1/2)\,[\max(k,l)]^{\beta}$, $k,l > \frac{1}{3\varepsilon}$, and $k-l \neq j-i$. Then $(\alpha_k \alpha_l)^{-1} \leq C/\gamma^2$.

Using that ω is γ-τ-Diophantine, we get

$$\Big|(\omega k - j) - (\omega l - i)\Big| = \Big|\omega(k-l) - (j-i)\Big| \geq \frac{\gamma}{|k-l|^{\tau}} \geq \frac{C\gamma}{[\max(k,l)]^{\beta\tau}}$$
$$\geq \frac{C}{2}\Big(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\Big)\,,$$

so that

$$|\omega k - j| \geq \frac{C\gamma}{2k^{\beta\tau}} \quad \text{or} \quad |\omega l - i| \geq \frac{C\gamma}{2l^{\beta\tau}}\,.$$

Assume, for instance, the first inequality. Therefore

$$|\omega^2 k^2 - j^2| = |\omega k - j|\,|\omega k + j| \geq C'\gamma\,k^{1-\beta\tau} = C'\gamma\,k^{\tau-1}\,,$$

since $\beta := (2-\tau)/\tau$. Hence, for ε small enough, $\alpha_k \geq C'\gamma\,k^{\tau-1}/2$, and estimating α_l with the worst possible lower bound (4.80), we deduce

$$\frac{1}{\alpha_k \alpha_l} \leq \frac{C l^{\tau-1}}{\gamma^2 k^{\tau-1}} \leq \frac{C'}{\gamma^2}$$

because $l \leq 2k$.

Collecting the estimates of all the previous cases, (4.85) follows. ∎

Remark 4.36. The analysis of the small divisors in Cases II–IV of the previous lemma corresponds, in the language of [51] and [48], to the property of "separation of the singular sites."

Lemma 4.37. (Bound of the off-diagonal operator $\mathcal{R}_1$) *Assume (4.25). There exists $\widetilde{C} > 0$ such that*

$$\left\| \mathcal{R}_1 h \right\|_{\sigma, s+\frac{\tau-1}{2}} \le \frac{\widetilde{C} C_1}{\varepsilon^{\frac{\tau-1}{2}} \gamma} \|h\|_{\sigma, s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)} . \tag{4.88}$$

Proof. We recall that

$$\mathcal{R}_1 h := |D|^{-1/2} P_n \Pi_W \Big(\bar{b}(t,x) \, |D|^{-1/2} h \Big) ,$$

so that for $h \in W^{(n)}$,

$$(\mathcal{R}_1 h)(t,x) = \sum_{|k| \le L_n} (\mathcal{R}_1 h)_k(x) \exp(\mathrm{i}kt)$$

with

$$\begin{aligned} (\mathcal{R}_1 h)_k &= |D_k|^{-1/2} \pi_k \Big(\bar{b} \, |D|^{-1/2} h \Big)_k \\ &= |D_k|^{-1/2} \pi_k \Big[\sum_{|l| \le L_n} b_{k-l} |D_l|^{-1/2} h_l \Big] . \end{aligned} \tag{4.89}$$

Set $B_m := \|b_m(x)\|_{H^1}$. From (4.89), (4.82), and since the zero time-Fourier coefficient of $\bar{b}$ is $B_0 = 0$,

$$\left\| (\mathcal{R}_1 h)_k \right\|_{H^1} \le C \sum_{|l| \le L_n, l \ne k} \frac{B_{k-l}}{\sqrt{\alpha_k} \sqrt{\alpha_l}} \|h_l\|_{H^1} . \tag{4.90}$$

Hence, by (4.85),

$$\left\| (\mathcal{R}_1 h)_k \right\|_{H^1} \le \frac{C}{\gamma \, \varepsilon^{\frac{\tau-1}{2}}} S_k , \quad \text{where} \quad S_k := \sum_{|l| \le L_n} B_{k-l} |k-l|^{\frac{\tau-1}{\beta}} \|h_l\|_{H^1} . \tag{4.91}$$

By (4.91), setting

$$\widetilde{S}(t) := \sum_{|k| \le L_n} S_k \exp(\mathrm{i}kt)$$

(with $S_{-k} = S_k$) and $s' := s + (\tau - 1)/2$,

$$\begin{aligned} \left\| \mathcal{R}_1 h \right\|^2_{\sigma, s'} &= \sum_{|k| \le L_n} \exp(2\sigma |k|)(k^{2s'} + 1) \left\| (\mathcal{R}_1 h)_k \right\|^2_{H^1} \\ &\le \frac{C^2}{\gamma^2 \varepsilon^{\tau-1}} \sum_{|k| \le L_n} \exp(2\sigma |k|)(k^{2s'} + 1) s_k^2 \\ &= \frac{C^2}{\gamma^2 \varepsilon^{\tau-1}} \|\widetilde{S}\|^2_{\sigma, s'} . \end{aligned} \tag{4.92}$$

It turns out that

$$\widetilde{S} = P_n(\widetilde{b}\widetilde{c}) \,,$$

where

$$\widetilde{b}(t) := \sum_{l\in\mathbf{Z}} |l|^{\frac{\tau-1}{\beta}} B_l \exp(\mathrm{i}lt) \quad \text{and} \quad \widetilde{c}(t) := \sum_{|l|\le L_n} \|h_l\|_{H^1} \exp(\mathrm{i}lt) \,.$$

Therefore, by (4.92) and since $s' > 1/2$,

$$\begin{aligned} \left\|\mathcal{R}_1 h\right\|_{\sigma,s'} &\le \frac{C}{\gamma\,\varepsilon^{\frac{\tau-1}{2}}}\|\widetilde{b}\widetilde{c}\|_{\sigma,s'} \le \frac{C}{\gamma\,\varepsilon^{\frac{\tau-1}{2}}}\|\widetilde{b}\|_{\sigma,s'}\|\widetilde{c}\|_{\sigma,s'} \\ &\le \frac{C}{\gamma\,\varepsilon^{\frac{\tau-1}{2}}}\|\bar{b}\|_{\sigma,s'+\frac{\tau-1}{\beta}}\|h\|_{\sigma,s'} \,, \end{aligned} \tag{4.93}$$

since $\|\widetilde{b}\|_{\sigma,s'} \le \|\bar{b}\|_{\sigma,s'+\frac{\tau-1}{\beta}}$ and $\|\widetilde{c}\|_{\sigma,s'} = \|h\|_{\sigma,s'}$.

Now, since $0 < \beta < 1$,

$$\|\bar{b}\|_{\sigma,s+\frac{\tau-1}{2}+\frac{\tau-1}{\beta}} \le \|\bar{b}\|_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le \|b\|_{\sigma,s+\frac{2\tau(\tau-1)}{2-\tau}} \le C_1 \tag{4.94}$$

by (4.25). Then (4.93) and (4.94) prove (4.88). ■

The estimate of $\mathcal{R}_2$ is derived by the regularizing property

$$\left\|\partial_w v_2\,[u]\right\|_{\sigma,s+2} \le C\|u\|_{\sigma,s} \tag{4.95}$$

of Proposition 4.5: by (4.83) the "loss of $\tau-1$ derivatives" due to $|D|^{-1/2}$ applied twice is compensated by the gain of two derivatives due to $\partial_w v_2$.

Lemma 4.38. (Estimate of $\mathcal{R}_2$) *Assume (4.25). There exists a constant $C > 0$ such that*

$$\left\|\mathcal{R}_2 h\right\|_{\sigma,s+\frac{\tau-1}{2}} \le \frac{CC_1}{\gamma}\|h\|_{\sigma,s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}.$$

Proof. Using that $\tau < 3$ to get (4.96), we have

$$\begin{aligned} \left\|\mathcal{R}_2 h\right\|_{\sigma,s+\frac{\tau-1}{2}} &\overset{(4.83)}{\le} \frac{C}{\sqrt{\gamma}}\left\|\mathcal{M}_2|D|^{-1/2}h\right\|_{\sigma,s+\tau-1} \\ &= \frac{C}{\sqrt{\gamma}}\left\|P_n\Pi_W\Big(b\,\partial_w v_2\Big[|D|^{-1/2}h\Big]\Big)\right\|_{\sigma,s+\tau-1} \\ &\le \frac{C}{\sqrt{\gamma}}\|b\|_{\sigma,s+\tau-1}\left\|\partial_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{\sigma,s+\tau-1} \\ &\le \frac{C'}{\sqrt{\gamma}}\|b\|_{\sigma,s+\tau-1}\left\|\partial_w v_2\Big[|D|^{-1/2}h\Big]\right\|_{\sigma,s+2} \qquad (4.96) \\ &\overset{(4.95)}{\le} \frac{C}{\sqrt{\gamma}}\|b\|_{\sigma,s+\tau-1}\left\||D|^{-1/2}h\right\|_{\sigma,s} \\ &\overset{(4.83)}{\le} \frac{CC_1}{\gamma}\|h\|_{\sigma,s+\frac{\tau-1}{2}} \,, \end{aligned}$$

since $\|b\|_{\sigma,s+\tau-1} \leq \|b\|_{\sigma,s+\frac{2\tau(\tau-1)}{2-\tau}} \leq C_1$ by (4.94). ∎

Proof of property (P3) *completed.* By Lemma 4.34 the operator U is invertible in $X_{\sigma,s+\frac{\tau-1}{2}}$, and by Lemmas 4.37 and 4.38, provided ε is small enough,

$$\varepsilon\left\|U^{-1}\mathcal{R}_1\right\|_{\mathcal{L}(X_{\sigma,s+(\tau-1)/2})}, \ \varepsilon\left\|U^{-1}\mathcal{R}_2\right\|_{\mathcal{L}(X_{\sigma,s+(\tau-1)/2})} < \frac{1}{4}.$$

Therefore also

$$U - \varepsilon\mathcal{R}_1 - \varepsilon\mathcal{R}_2 \colon X_{\sigma,s+\frac{\tau-1}{2}} \to X_{\sigma,s+\frac{\tau-1}{2}}$$

has a bounded inverse:

$$\begin{aligned}\left\|(U - \varepsilon\mathcal{R}_1 - \varepsilon\mathcal{R}_2)^{-1}h\right\|_{\sigma,s+\frac{\tau-1}{2}} &= \left\|(I - \varepsilon U^{-1}\mathcal{R}_1 - \varepsilon U^{-1}\mathcal{R}_2)^{-1}U^{-1}h\right\|_{\sigma,s+\frac{\tau-1}{2}} \\ &\leq 2\|U^{-1}h\|_{\sigma,s+\frac{\tau-1}{2}} \leq C\|h\|_{\sigma,s+\frac{\tau-1}{2}}. \qquad (4.97)\end{aligned}$$

We finally deduce

$$\begin{aligned}\left\|\mathcal{L}_n^{-1}h\right\|_{\sigma,s} &= \left\||D|^{-1/2}(U - \varepsilon\mathcal{R}_1 - \varepsilon\mathcal{R}_2)^{-1}|D|^{-1/2}h\right\|_{\sigma,s} \\ &\overset{(4.83)}{\leq} \frac{C}{\sqrt{\gamma}}\left\|(U - \varepsilon\mathcal{R}_1 - \varepsilon\mathcal{R}_2)^{-1}|D|^{-1/2}h\right\|_{\sigma,s+\frac{\tau-1}{2}} \\ &\overset{(4.97)}{\leq} \frac{C'}{\sqrt{\gamma}}\left\||D|^{-1/2}h\right\|_{\sigma,s+\frac{\tau-1}{2}} \overset{(4.83)}{\leq} \frac{C''}{\gamma}\|h\|_{\sigma,s+\tau-1} \\ &\leq \frac{C''}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s}\end{aligned}$$

because $h \in W^{(n)}$. This completes the proof of property (P3). ∎

5

Forced Vibrations

We discuss in this chapter the problem of *forced* vibrations: we look for nontrivial periodic solutions of the equation

$$\begin{cases} u_{tt} - u_{xx} = \varepsilon f(t, x, u) \, , \\ u(t, 0) = u(t, \pi) = 0 \, , \end{cases} \tag{5.1}$$

where the nonlinearity is T-periodic in time:

$$f(t + T, x, u) = f(t, x, u) \, . \tag{5.2}$$

We look for T-periodic solutions.

It is clear that the bifurcation problem will be drastically different according to whether the forcing frequency $\omega := 2\pi/T$ satisfies either

(i) $\omega \in \mathbf{Q}$

or

(ii) $\omega \in \mathbf{R} \setminus \mathbf{Q}$.

This second case (ii) leads to a small-divisor problem similar to the one discussed in the previous chapters. Some existence results in this direction have been obtained in [107] (for ω given and ε small enough) and, more recently, in [62] and [12] also for $\varepsilon = 1$ and ω large (rapid vibrations).

We shall now be concerned with the case that the forcing frequency ω is a rational number, i.e., with case (i).

5.1 The Forcing Frequency $\omega \in \mathbf{Q}$

For simplicity of exposition we shall assume

$$T = 2\pi \, , \qquad \text{i.e.,} \quad \omega = 1 \, , \tag{5.3}$$

and so we look for nontrivial 2π-periodic solutions of (5.1).

The spectrum of the D'Alembert operator in the space of 2π-periodic in time functions with spatial Dirichlet boundary conditions is

$$\sigma(\partial_{tt} - \partial_{xx}) = \Big\{ -l^2 + j^2 \mid l \in \mathbf{Z},\ j \in \mathbf{N} \Big\} \subset \mathbf{Z}, \tag{5.4}$$

and it is therefore formed by integers.

Zero is an eigenvalue of infinite multiplicity (when $|l| = j$), but the spectrum is NOT dense in **R**, and in particular, the other eigenvalues are well separated from 0 if $|l| \neq j$. This is the big difference with respect to case (*ii*) when ω is an irrational number.

For these reasons the main difficulty for proving existence of periodic solutions of (5.1) in case (*i*) does not lie in solving the range equation, but in solving the bifurcation equation, which has an intrinsic *lack of compactness* (see Remark 5.8).

The first breakthrough regarding problem (5.1)–(5.3) was achieved by Rabinowitz in [114], where, using methods inspired by the theory of elliptic regularity, existence and regularity of solutions was proved for nonlinearities satisfying the strong monotonicity assumption

$$(\partial_u f)(t, x, u) \geq \beta > 0\,. \tag{5.5}$$

Theorem 5.1. (Rabinowitz [114]) *Let $f \in C^k$ satisfy (5.2), (5.3), (5.5). Then $\forall \varepsilon$ small enough, there exists a 2π-periodic solution $u \in H^k$ of (5.1).*

Other existence results of weak and classical solutions have been obtained, still for strongly monotone nonlinearities, in [89], [55], [43].

Subsequently, Rabinowitz [115] was able to prove existence of weak solutions (actually a continuum branch) for a class of weakly monotone nonlinearities such as

$$f(t, x, u) = u^{2k+1} + G(t, x, u)\,,$$

where

$$G(t, x, u_2) \geq G(t, x, u_1) \quad \text{if} \quad u_2 \geq u_1\,.$$

The weak solutions obtained in [115] are only continuous functions. In general, more regularity is not expected: Brezis and Nirenberg [43] proved—but only for strictly monotone nonlinearities—that any 2π-periodic L^∞-solution of (5.1) is smooth, even in the nonperturbative case $\varepsilon = 1$, whenever the nonlinearity f is smooth (remark the difference with the autonomous case discussed in the previous chapter where actually analytic solutions were found).

The monotonicity assumption (strong or weak) on the nonlinearity is the key property in [114]–[115] for overcoming the lack of compactness in the infinite-dimensional bifurcation equation; see Section 5.3.

On the other hand, little is known about existence and regularity of solutions without the monotonicity of f.

The first existence results of weak solutions for nonmonotone forcing terms were obtained by Willem [131], Hofer [71], and Coron [47] for nonlinearities like

$$f(t,x,u) = g(u) + h(t,x)\,,$$

where $g(u)$ satisfies suitable linear growth conditions and $\varepsilon = 1$ (nonperturbative case).

Existence of weak solutions is proved, in [131], [71], for a set of h dense in L^2, although explicit criteria that characterize such h are not provided. The infinite-dimensional bifurcation problem is overcome by assuming nonresonance hypotheses between the asymptotic behavior of $g(u)$ and the spectrum of the D'Alembert operator.

On the other hand, Coron [47] finds weak solutions assuming the additional symmetry $h(t,x) = h(t+\pi, \pi - x)$ and restricting to the space of functions satisfying $u(t,x) = u(t+\pi, \pi - x)$, where the kernel of the D'Alembert operator reduces to 0.

5.1. Exercise: Prove the following statement: the unique solution of the linear homogeneous wave equation (2.8) satisfying the symmetry condition

$$u(t,x) = u(t+\pi, \pi - x) \tag{5.6}$$

is $u = 0$.

More recently, existence results of periodic solutions for concave and convex nonlinearities have been obtained in [17].

In [21]–[22] existence and regularity of solutions of (5.1) have been proved for a large class of nonmonotone forcing terms $f(t,x,u)$, including, for example,

$$f(t,x,u) = \pm u^{2k} + h(t,x)$$

and $h > 0$; see Section 5.4.

In this chapter we first prove the Rabinowitz theorem, Theorem 5.1; see Section 5.2. Next we prove some existence results of forced vibrations for nonmonotone nonlinearities extracted from [22].

5.2 The Variational Lyapunov–Schmidt Reduction

In view of the variational argument that we shall use to solve the bifurcation equation, we look for solutions

$$u\colon \Omega := \mathbf{T} \times (0,\pi) \to \mathbf{R}$$

of (5.1) in the Banach space

$$E := H^1(\Omega) \cap C_0^{1/2}(\bar{\Omega})\,,$$

where $H^1(\Omega)$ is the usual Sobolev space and $C_0^{1/2}(\bar{\Omega})$ is the space of all the 1/2-Hölder continuous functions $u\colon \bar{\Omega} \to \mathbf{R}$ satisfying

$$u(t,0) = u(t,\pi) = 0$$

endowed with norm

$$\|u\|_E := \|u\|_{H^1(\Omega)} + \|u\|_{C^{1/2}(\bar{\Omega})} ,$$

where

$$\|u\|^2_{H^1(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Omega)}$$

and

$$\|u\|_{C^{1/2}(\bar{\Omega})} := \|u\|_{C^0(\Omega)} + \sup_{(t,x)\neq(t_1,x_1)} \frac{|u(t,x) - u(t_1,x_1)|}{(|t-t_1| + |x-x_1|)^{1/2}} .$$

Remark 5.2. The choice of the space E is motivated by the regularizing property of the inverse of the D'Alembert operator proved in Lemma 5.4.

Critical points of the Lagrangian action functional $\Psi \in C^1(E, \mathbf{R})$,

$$\Psi(u) := \Psi(u,\varepsilon) := \int_\Omega \Big[\frac{u_t^2}{2} - \frac{u_x^2}{2} + \varepsilon F(t,x,u)\Big]\, \mathrm{d}t\mathrm{d}x ,$$

where

$$F(t,x,u) := \int_0^u f(t,x,\xi)\mathrm{d}\xi ,$$

are weak solutions of (5.1).

For $\varepsilon = 0$, the critical points of Ψ in E reduce to the solutions of the linear equation (2.8) and form the space

$$V := N \cap H^1(\Omega) \subset E , \tag{5.7}$$

where

$$N := \Big\{ v(t,x) = \hat{v}(t+x) - \hat{v}(t-x) =: v_+(t,x) - v_-(t,x) \\ \text{such that } \hat{v} \in L^2(\mathbf{T}) \text{ and } \int_0^{2\pi} \hat{v}(s)\, \mathrm{d}s = 0 \Big\}$$

is the L^2-closure of the classical solutions of the linear equation (2.8).

We have $V \subset E$ because any function $\hat{v} \in H^1(\mathbf{T})$ is $1/2$-Hölder continuous.

Remark 5.3. The only difference between the space V introduced in (2.14) with respect to V defined in (5.7) is that the functions v in (2.14) are not necessarily even in time. For notational convenience we denote these spaces in the same way.

Decompose

$$E = V \oplus W ,$$

where

$$W := N^\perp \cap E$$

and

$$N^{\perp} := \left\{ h \in L^2(\Omega) \mid \int_{\Omega} hv = 0, \ \forall v \in N \right\}$$
$$= \left\{ h = \sum_{j \geq 1, j \neq |l|} h_{lj} e^{ilt} \sin jx \ \text{ with } \ \|h\|^2_{L^2} = \pi^2 \sum |h_{lj}|^2 < \infty \right\}.$$

Projecting (5.1), for $u = v + w$ with $v \in V$, $w \in W$, yields

$$\begin{cases} 0 = \Pi_N f(v + w) & \textit{bifurcation equation}, \\ w_{tt} - w_{xx} = \varepsilon \Pi_{N^\perp} f(v + w) & \textit{range equation}, \end{cases}$$

where Π_N and $\Pi_{N^\perp}$ are the projectors from $L^2(\Omega)$ onto N and $N^\perp$, and $f(u)$ denotes the Nemitski operator induced on E by the nonlinearity

$$f(u)(t, x) := f(t, x, u).$$

In case the nonlinearity is strictly monotone, the usual approach of [114], [55], and [6] is to solve first the bifurcation equation, finding its unique solution $v = v(w)$, and next, to solve the range equation.

In contrast, for a nonmonotone nonlinearity the bifurcation equation can *not* in general be solved uniquely (see Remark 5.15). Therefore, we solve first the range equation and thereafter the bifurcation equation.

In the case of monotone f this approach also makes some technical aspect of the proof easier than in [114].

5.2.1 The Range Equation

We first study the invertibility properties of the D'Alembert operator $\Box := \partial_{tt} - \partial_{xx}$ restricted to $N^\perp$.

Lemma 5.4. *The inverse of the D'Alembert operator $\Box^{-1}: N^\perp \to W$ defined by*

$$\Box^{-1} f := \sum_{j \geq 1, j \neq |l|} \frac{f_{lj}}{-l^2 + j^2} e^{ilt} \sin jx, \qquad \forall f \in N^\perp, \tag{5.8}$$

is a bounded operator, i.e., satisfies

$$\left\| \Box^{-1} f \right\|_E \leq C \, \|f\|_{L^2} \tag{5.9}$$

for a suitable positive constant C.

Proof. Let $h := \Box^{-1} f$. For any fixed $x \in (0, \pi)$ we have

$$\left\| \partial_t h(\cdot, x) \right\|^2_{L^2(\mathbf{T})} = 2\pi \sum_{l \in \mathbf{Z}} l^2 |h_l(x)|^2, \tag{5.10}$$

where

$$h_l(x) := \sum_{j\neq|l|} \frac{f_{lj}}{-l^2+j^2} \sin jx$$

satisfies

$$|h_l(x)|^2 \leq \Big(\sum_{j\neq|l|} \frac{|f_{lj}|}{|j^2-l^2|}\Big)^2 \leq \sum_{j\neq|l|} |f_{lj}|^2 \sum_{j\neq|l|} \frac{1}{|j^2-l^2|^2} \tag{5.11}$$

by Cauchy–Schwarz. By (5.10)–(5.11),

$$\Big\|\partial_t h(\cdot,x)\Big\|^2_{L^2(\mathbf{T})} \leq 2\pi \sum_{l\in\mathbf{Z}} \Big(\sum_{j\neq|l|} |f_{lj}|^2\Big)\Big(\sum_{j\neq|l|} \frac{l^2}{|j^2-l^2|^2}\Big). \tag{5.12}$$

Now

$$\frac{l}{j^2-l^2} = \frac{1}{2(j-l)} - \frac{1}{2(j+l)},$$

whence for any l,

$$\sum_{j\neq|l|} \frac{l^2}{|l^2-j^2|^2} \leq \sum_{j\neq|l|} \frac{1}{(j-l)^2} + \frac{1}{(j+l)^2} \leq 4\sum_{j\geq 1} \frac{1}{j^2} =: M^2,$$

and by (5.12),

$$\Big\|\partial_t h(\cdot,x)\Big\|^2_{L^2(\mathbf{T})} \leq 2\pi M^2 \sum_{l\in\mathbf{Z}} \sum_{j\neq|l|} |f_{lj}|^2 = \frac{2M^2}{\pi} \|f\|^2_{L^2(\Omega)}. \tag{5.13}$$

As a consequence, for any fixed x, by the Hölder inequality,

$$\begin{aligned} |h(t,x)-h(t',x)| &\leq \Big(\int_t^{t'} |\partial_t h(\tau,x)|^2\,\mathrm{d}\tau\Big)^{1/2}\sqrt{|t-t'|} \\ &\overset{(5.13)}{\leq} M\sqrt{2\pi^{-1}}\|f\|_{L^2(\Omega)}\sqrt{|t-t'|} \end{aligned} \tag{5.14}$$

and $h(\cdot,x)$ is $1/2$-Hölder continuous, uniformly in x.

Similarly, for any fixed t,

$$\Big\|\partial_x h(t,\cdot)\Big\|^2_{L^2(0,\pi)} \leq \frac{M^2}{2\pi}\|f\|^2_{L^2(\Omega)} \tag{5.15}$$

and so

$$|h(t,x')-h(t,x)| \leq \frac{M}{\sqrt{2\pi}}\|f\|_{L^2(\Omega)}\sqrt{|x-x'|} \tag{5.16}$$

uniformly in t. By (5.14) and (5.16) we deduce

$$\|h\|_{C^{1/2}(\Omega)} \leq C\|f\|_{L^2},$$

and integrating (5.13) and (5.15) we also get

$$\|\partial_t h\|^2_{L^2(\Omega)}, \|\partial_x h\|^2_{L^2(\Omega)} \leq C'\|f\|^2_{L^2(\Omega)},$$

implying (5.9). ∎

Remark 5.5. The previous lemma could also be proved via the integral representation formula for $\Box^{-1}$ given in [89]:

$$\Box^{-1} f = -\frac{1}{2}\int_x^{\pi}\int_{t+x-\xi}^{t-x+\xi} f(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi + c\frac{(\pi - x)}{\pi}, \tag{5.17}$$

where

$$c := \frac{1}{2}\int_0^{\pi}\int_{t-\xi}^{t+\xi} f(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \equiv \frac{1}{2}\int_0^{\pi}\int_{-\xi}^{\xi} f(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \equiv \text{const}$$

is a constant independent of t (because $f \in N^{\perp}$).

Using (5.17), it follows that $\Box^{-1}$ is a bounded operator also between the spaces

$$H^k(\Omega) \longrightarrow H^{k+1}(\Omega), \quad C^k(\bar{\Omega}) \longrightarrow C^{k+1}(\bar{\Omega}). \tag{5.18}$$

Fixed points $w \in W$ of

$$w = \varepsilon\Box^{-1}\Pi_{N^{\perp}} f(v+w)$$

are solutions of the range equation.

Applying the contraction mapping theorem as in Lemma 2.11 we have the following lemma:

Lemma 5.6. (Solution of the range equation) $\forall\, R > 0$, $\exists\, \varepsilon_0(R) > 0$, $C_0(R) > 0$, *such that*

$$\forall\, |\varepsilon| \le \varepsilon_0(R) \quad \textit{and} \quad \forall\, v \in N \textit{ with } \|v\|_{L^\infty} \le 2R$$

there exists a unique solution $w(\varepsilon, v) \in W$ *of the range equation satisfying*

$$\|w(\varepsilon, v)\|_E \le C_0(R)|\varepsilon|. \tag{5.19}$$

Moreover, the map $(\varepsilon, v) \to w(\varepsilon, v)$ *is* $C^1(\{\|v\|_{L^\infty} \le 2R\}, W)$.

5.2. Exercise: Prove the existence, for ε small enough, of a unique 2π-periodic solution in E of

$$\begin{cases} u_{tt} - u_{xx} = \varepsilon(f(u) + \cos(2t)\sin(x)), \\ u(t,0) = u(t,\pi) = 0. \end{cases}$$

5.2.2 The Bifurcation Equation

Once the range equation has been solved by $w(\varepsilon, v) \in W$, there remains the infinite-dimensional bifurcation equation

$$\Pi_N\, f(v + w(\varepsilon, v)) = 0,$$

which is equivalent, since V is dense in N with the L^2-norm, to

$$\int_\Omega f(v + w(\varepsilon, v))\phi = 0 \qquad \forall \phi \in V\,. \tag{5.20}$$

By the variational Lyapunov–Schmidt reduction (see the same arguments in Section 2.3), (5.20) is the Euler–Lagrange equation of the *reduced Lagrangian action functional*[1]

$$\Phi\colon \left\{ \|v\|_{H^1} < 2R \right\} \to \mathbf{R}$$

defined by

$$\Phi(v) := \Phi(\varepsilon, v) := \Psi(v + w(\varepsilon, v), \varepsilon)\,,$$

which can be written (as in (2.29) with $\omega = 1$) as

$$\Phi(v) = \varepsilon \int_\Omega \Big[F(v + w(\varepsilon, v)) - \frac{1}{2} f(v + w(\varepsilon, v))w(\varepsilon, v) \Big]\, \mathrm{d}t\mathrm{d}x\,. \tag{5.21}$$

Lemma 5.7. *The following continuity property holds:*

$$\|v_n\|_{H^1}, \|\bar v\|_{H^1} \le R\,, \quad v_n \stackrel{L^\infty}{\longrightarrow} \bar v \quad \Longrightarrow \quad \Phi(v_n) \longrightarrow \Phi(\bar v)\,. \tag{5.22}$$

Proof. Setting $w_n := w(\varepsilon, v_n)$ and $\bar w := w(\varepsilon, \bar v)$, we have

$$\begin{aligned} \left| \int_\Omega F(v_n + w_n) - F(\bar v + \bar w) \right| &\le \max_{\bar\Omega \times \{|u| \le R+1\}} |f(t, x, u)| \int_\Omega |v_n - \bar v + w_n - \bar w| \\ &\le C(R)\Big(\|v_n - \bar v\|_{L^1} + \|w_n - \bar w\|_{L^1} \Big) \longrightarrow 0 \end{aligned}$$

as $n \to \infty$, by the fact that $\|v_n - \bar v\|_{L^\infty} \to 0$ and therefore, by Lemma 5.6, $\|w(\varepsilon, v_n) - w(\varepsilon, \bar v)\|_E \to 0$. An analogous estimate holds for the second term in the integral (5.21). ∎

The functional Φ lacks compactness properties, and to find its critical points we cannot rely on critical point theory.

Remark 5.8. In the corresponding reduced Lagrangian action functional in (2.29) for the autonomous case, it is the "elliptic" term $\|v\|^2_{H^1}$ that introduces compactness in the problem, allowing one to find mountain-pass critical points.

In the case that f is strictly monotone, it is natural to try to minimize Φ, because F is strictly convex in u.

However, we do not apply the direct methods of the calculus of variations (see Appendix B.2) because without assuming any growth condition on the nonlinearity f, the functional Φ could not be well defined on any L^p-space.

Following [114] we make a constrained minimization:

[1] Since $\|v\|_{L^\infty(\Omega)} \le \|v\|_{H^1(\Omega)}$, the functional $\Phi(v)$ is well defined for any $\|v\|_{H^1(\Omega)} < 2R$ by Lemma 5.6.

Lemma 5.9. $\forall R > 0$, *the functional* Φ *attains a minimum in*

$$\bar{B}_R := \left\{ v \in V, \|v\|_{H^1} \leq R \right\}.$$

Proof. Let $v_n \in \bar{B}_R$ be a minimizing sequence $\Phi(v_n) \to \inf_{\bar{B}_R} \Phi$. Since v_n is bounded in V, up to a subsequence $v_n \overset{H^1}{\rightharpoonup} \bar{v}$, and by the compact embedding (2.15), $v_n \overset{L^\infty}{\longrightarrow} \bar{v}$. Therefore by (5.22), $\bar{v}$ is a minimum point of Φ restricted to $\bar{B}_R$. ∎

Since $\bar{v}$ could belong to the boundary $\partial \bar{B}_R$, we only have the variational inequality

$$D_v \Phi(\bar{v})\,[\phi] = \int_\Omega f(\bar{v} + w(\varepsilon, \bar{v}))\phi \leq 0 \tag{5.23}$$

for any *admissible variation* $\phi \in V$, namely for any $\phi \in V$ such that

$$\bar{v} + \theta\phi \in B_R\,, \quad \forall \theta < 0$$

sufficiently small; see Figure 5.1.

A sufficient condition for $\phi \in V$ to be an admissible variation is

$$\langle \bar{v}, \phi \rangle_{H^1} > 0 \tag{5.24}$$

because

$$\begin{aligned} \|\bar{v} + \theta\phi\|^2_{H^1} &= \|\bar{v}\|^2_{H^1} + 2\theta \langle \bar{v}, \phi \rangle_{H^1} + \theta^2 \|\phi\|^2_{H^1} \\ &\leq R^2 + 2\theta \langle \bar{v}, \phi \rangle_{H^1} + \theta^2 \|\phi\|^2_{H^1} < R^2 \end{aligned}$$

for $\theta < 0$ small enough.

The heart of the existence proof of Theorem 5.1 is to obtain, choosing suitable admissible variations, the *a priori estimate* $\|\bar{v}\|_{H^1} < R_*$ for some $R_* > 0$ (independent of ε), i.e., to show that $\bar{v}$ is an *inner* minimum point of Φ in B_{R_*}.

It is here where the monotonicity plays a role.

5.3 Monotone f

In the sequel κ_i will denote positive constants independent of ε (possibly depending on the nonlinearity).

5.3.1 Step 1: the L^∞ Estimate

Using the variational inequality (5.23) and (5.19),

$$\begin{aligned} \int_\Omega f(\bar{v})\phi &= \int_\Omega f(\bar{v} + w(\varepsilon, \bar{v}))\phi + \int_\Omega [f(\bar{v}) - f(\bar{v} + w(\varepsilon, \bar{v}))]\phi \\ &\overset{(5.23)}{\leq} \int_\Omega [f(\bar{v}) - f(\bar{v} + w(\varepsilon, \bar{v}))]\phi \overset{(5.19)}{\leq} |\varepsilon| C_1(R) \|\phi\|_{L^1}\,, \end{aligned} \tag{5.25}$$

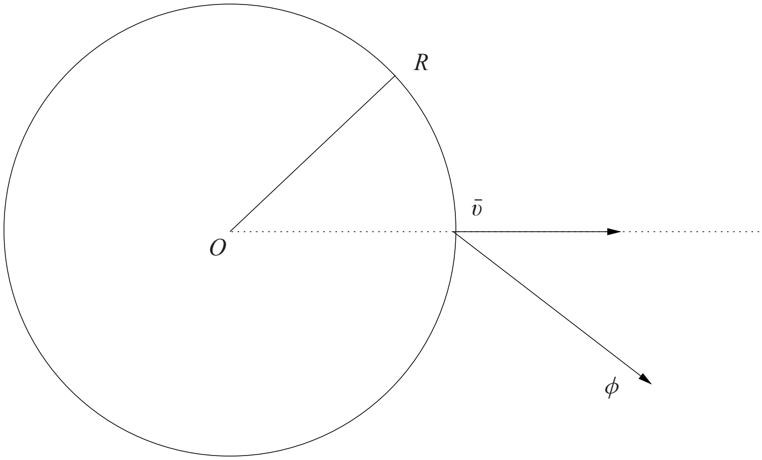

Fig. 5.1. The variational inequality.

where $C_1(\cdot)$ is a suitable increasing function depending on f.

Then there exists a decreasing function $0 < \varepsilon_1(\cdot) \leq \varepsilon_0(\cdot)$ such that

$$\int_\Omega f(\bar{v})\phi \leq \|\phi\|_{L^1} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1(R)\,. \tag{5.26}$$

We now choose

$$\phi = q(\bar{v}_+) - q(\bar{v}_-) = q(\hat{\bar{v}}(t+x)) - q(\hat{\bar{v}}(t-x)) \in V\,,$$

where

$$q(\lambda) := q_M(\lambda) := \begin{cases} 0\,, & \text{if} \quad |\lambda| \leq M\,, \\ \lambda - M\,, & \text{if} \quad \lambda \geq M\,, \\ \lambda + M\,, & \text{if} \quad \lambda \leq M\,, \end{cases}$$

and[2]

$$M := \frac{1}{2}\|\hat{\bar{v}}\|_{L^\infty(\mathbf{T})}\,. \tag{5.27}$$

We can assume $M > 0$, i.e., $\bar{v}$ is not identically zero.

Lemma 5.10. ([114]) *$\phi \in V$ is an admissible variation and*

$$\bar{v}(t,x)\phi(t,x) \geq 0\,, \qquad \forall (t,x) \in \Omega\,. \tag{5.28}$$

Proof. By Lemma 2.20, for $p, q \in L^1(\mathbf{T})$,

$$\int_\Omega p(t+x)q(t-x)\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} p(s)\,\mathrm{d}s \int_0^{2\pi} q(s)\,\mathrm{d}s\,. \tag{5.29}$$

[2] Note that $\|\hat{v}\|_{L^\infty(\mathbf{T})} \leq \|v\|_{L^\infty(\Omega)} \leq 2\|\hat{v}\|_{L^\infty(\mathbf{T})}$, $\forall v \in N \cap L^\infty$.

Using repeatedly (5.29) and since $\hat{\bar{v}}$ has zero average,

$$\begin{aligned}\langle \bar{v}, \phi\rangle_{H^1} = \langle \bar{v}_+ - \bar{v}_-, q(\bar{v}_+) - q(\bar{v}_-)\rangle_{H^1} &= \int_\Omega \bar{v}_+ q(\bar{v}_+) + \bar{v}_- q(\bar{v}_-) \\ &\quad + 2\int_\Omega q'(\bar{v}_+)\Big[(\bar{v}_+)_t^2 + (\bar{v}_+)_x^2\Big] \\ &> 0\end{aligned}$$

because $q' \geq 0$ and $\bar{v}_\pm q(\bar{v}_\pm) > 0$ on a set of positive measure, since q is a monotone odd function and by our choice of M.

Finally, since q is a monotone function,

$$\bar{v}\phi = (\bar{v}_+ - \bar{v}_-)(q(\bar{v}_+) - q(\bar{v}_-)) \geq 0\,,$$

proving (5.28). ■

We now estimate from below the left-hand side in (5.25). By the mean value theorem, the strong monotonicity assumption (5.5), and (5.28),

$$\begin{aligned}\int_\Omega f(\bar{v})\phi &= \int_\Omega \Big[f(t,x,0) + f_u(\text{intermediate point})\bar{v}\Big]\phi \\ &\geq \int_\Omega f(t,x,0)\phi + \beta\int_\Omega \bar{v}\phi\,,\end{aligned}$$

whence

$$\begin{aligned}\beta\int_\Omega \bar{v}\phi &\leq \int_\Omega |f(t,x,0)||\phi| + \int_\Omega f(\bar{v})\phi \\ &\overset{(5.26)}{\leq} \max|f(t,x,0)|\|\phi\|_{L^1} + \|\phi\|_{L^1} := \kappa_1\|\phi\|_{L^1}\,. \qquad (5.30)\end{aligned}$$

Now, since $\hat{\bar{v}}$ has zero average, using (5.29) and again Lemma 2.20,

$$\begin{aligned}\int_\Omega \bar{v}\phi = \int_\Omega \bar{v}_+ q(v_+) + \bar{v}_- q(v_-) &\overset{(2.56)}{=} \pi\int_0^{2\pi} \hat{\bar{v}}(s) q(\hat{\bar{v}}(s))\,\mathrm{d}s \\ &\geq \pi M\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})}\,, \qquad (5.31)\end{aligned}$$

since $\lambda q(\lambda) \geq M|q(\lambda)|$. Next,

$$\int_\Omega |\phi| \leq \int_\Omega |q(v_+)| + |q(v_-)| \overset{(2.56)}{=} 2\pi\,\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})}\,, \qquad (5.32)$$

and (5.31)–(5.32) imply

$$\int_\Omega \bar{v}\phi \geq \frac{1}{2}M\|\phi\|_{L^1} \overset{(5.27)}{=} \frac{1}{4}\|\bar{v}\|_{L^\infty}\|\phi\|_{L^1}\,. \qquad (5.33)$$

By (5.30) and (5.33) we finally deduce

$$\|\bar{v}\|_{L^\infty} \leq \frac{4\kappa_1}{\beta} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1(R)\,. \qquad (5.34)$$

5.3.2 Step 2: the H^1 Estimate

The H^1 estimate is carried out by inserting in the variational inequality (5.23) the variation

$$\phi := -D_{-h}D_h\bar{v}\,,$$

where

$$(D_h f)(t,x) := \frac{f(t+h,x)-f(t,x)}{h}$$

is the difference quotient with respect to t. Note that ϕ is *admissible* because

$$\langle -D_{-h}D_h\bar{v}, \bar{v}\rangle_{H^1} = \langle D_h\bar{v}, D_h\bar{v}\rangle_{H^1} > 0$$

by the formula for integration by parts

$$\int_\Omega f(D_{-h}g) = -\int_\Omega (D_h f)g\ , \qquad \forall f,g \in L^2(\Omega)\,. \tag{5.35}$$

We shall use the following technical lemma.

Lemma 5.11. *Let $f \in L^2(\Omega)$ have a weak derivative $f_t \in L^2(\Omega)$. Then*

$$\|D_h f\|_{L^2(\Omega)} \le \|f_t\|_{L^2(\Omega)} \tag{5.36}$$

and

$$D_h f \overset{L^2}{\to} f_t \quad \text{as} \quad h \to 0\,. \tag{5.37}$$

Proof. To prove (5.36) assume temporarily that f is smooth. From the fundamental theorem of calculus,

$$(D_h f)(t,x) = \frac{f(t+h,x)-f(t,x)}{h} = \int_0^1 f_t(t+hs,x)\,\mathrm{d}s\ .$$

By the Cauchy–Schwarz inequality, Fubini's theorem, and periodicity,

$$\begin{aligned}
\int_\Omega |D_h f(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x &= \int_0^\pi\int_0^{2\pi}\left|\int_0^1 f_t(t+hs,x)\,\mathrm{d}s\right|^2\mathrm{d}t\,\mathrm{d}x\\
&\le \int_0^\pi\int_0^{2\pi}\int_0^1 |f_t(t+hs,x)|^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_0^\pi\int_0^1\int_0^{2\pi} |f_t(t+hs,x)|^2\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\\
&= \int_0^\pi\int_0^1\int_0^{2\pi} |f_t(t,x)|^2\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\\
&= \int_0^1\int_0^\pi\int_0^{2\pi} |f_t(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s = \|f_t\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

Inequality (5.36) is valid for any f having a weak derivative $f_t \in L^2(\Omega)$ by approximation.

Proof of (5.37). We first show the weak L^2-convergence. Let $\phi \in C^1(\Omega)$. By (5.35) and the Lebesgue dominated convergence theorem,

$$\int_\Omega (D_h f)\phi = -\int_\Omega f(D_{-h}\phi) \overset{h\to 0}{\longrightarrow} -\int_\Omega f\phi_t = \int_\Omega f_t\phi\,, \tag{5.38}$$

since f has a weak derivative f_t.

Since by (5.36), $D_h f$ is bounded in L^2, and (5.38) holds in the dense subset $C^1(\Omega) \subset L^2(\Omega)$, we conclude that $D_h f \overset{L^2}{\rightharpoonup} f_t$.

By the weak lower semicontinuity of the norm and (5.36),

$$\|f_t\|_{L^2(\Omega)} \le \liminf \|D_h f\|_{L^2(\Omega)} \overset{(5.36)}{\le} \|f_t\|_{L^2(\Omega)}\,,$$

implying

$$\lim \|D_h f\|_{L^2(\Omega)} = \|f_t\|_{L^2(\Omega)}\,. \tag{5.39}$$

The weak convergence $D_h f \overset{L^2}{\rightharpoonup} f_t$ and (5.39) imply the strong convergence (5.37). ■

By the variational inequality (5.23), setting $\bar{w} := w(\varepsilon, \bar{v})$,

$$\begin{aligned} 0 &\overset{(5.23)}{\ge} \int_\Omega f(\bar{v}+\bar{w})\phi \\ &\overset{(5.35)}{=} \int_\Omega D_h f(\bar{v}+\bar{w}) D_h \bar{v} \\ &\overset{h\to 0}{\to} \int_\Omega \partial_t\Big(f(\bar{v}+\bar{w})\Big)\bar{v}_t \\ &= \int_\Omega \Big(f_t(\bar{v}+\bar{w}) + f_u(\bar{v}+\bar{w})(\bar{v}_t+\bar{w}_t)\Big)\bar{v}_t\,, \end{aligned} \tag{5.40}$$

where we have applied Lemma 5.11 (since $\bar{v} \in H^1$ and $\bar{w} \in E$).

By (5.40), using the L^∞-estimate (5.34) for $\bar{v}$ (which is independent of R) and (5.19), we derive

$$\int_\Omega f_u(\bar{v}+\bar{w})\bar{v}_t^2 \le \left|\int_\Omega \Big(f_t(\bar{v}+\bar{w}) + f_u(\bar{v}+\bar{w})\bar{w}_t\Big)\bar{v}_t\right| \le \kappa_2 \|\bar{v}_t\|_{L^2} \tag{5.41}$$

for $|\varepsilon| \le \varepsilon_2(R) \le \varepsilon_1(R)$.

By the strong monotonicity assumption $f_u \ge \beta > 0$,

$$\int_\Omega f_u(\bar{v}+\bar{w})\bar{v}_t^2 \ge \int_\Omega \beta \bar{v}_t^2\,,$$

implying, from (5.41), $\|\bar{v}_t\|_{L^2} < \kappa_2/\beta$ and, in conclusion,

$$\|\bar{v}\|_{H^1} < \kappa_3\,, \qquad \forall\, |\varepsilon| \le \varepsilon_2(R)\,.$$

Proof of Theorem 5.1 *completed.* Take

$$R_* := \kappa_3 \quad \text{and} \quad \varepsilon_* := \varepsilon_2(R_*)\,.$$

Therefore, $\forall\, |\varepsilon| \le \varepsilon_*$,

$$\|\bar{v}(\varepsilon)\|_{H^1} < R_*$$

is an interior minimum point of Φ in $B_{R_*} := \{\|v\|_{H^1} < R_*\}$ and

$$u := \bar{v}(\varepsilon) + w(\varepsilon, \bar{v}(\varepsilon)) \in E$$

is a weak solution of (5.1).

Using similar techniques (inspired by regularity theory), inserting suitable variations in the Euler–Lagrange equation

$$\int_\Omega f(\bar{v} + w(\varepsilon, \bar{v}))\phi = 0\,, \qquad \forall\, \phi \in V\,,$$

and exploiting the regularizing property (5.18) of $\square^{-1}$, one could obtain L^∞ and H^1 estimates for the higher-order derivatives of $\bar{v}$ and $w(\varepsilon, \bar{v})$, proving the higher regularity of the solution u; see [114] and [22].

We leave as an exercise the proof of the uniqueness of the solution u (in a ball of fixed radius independent of ε), which is a consequence of the monotonicity of f with respect to u. ■

5.4 Nonmonotone f

In this section we prove the following existence theorem of forced vibrations of (5.1) for nonmonotone nonlinearities.

Theorem 5.12. ([22]) *Let*

$$f(t, x, u) = u^{2k} + h(t, x)\,, \tag{5.42}$$

where $h \in N^\perp$ satisfies $h(t, x) > 0$ (or $h(t, x) < 0$) a.e. in Ω.

(i) (Existence) *For all ε small enough, there exists at least one weak solution $u \in E$ of (5.1) with $\|u\|_E \le C|\varepsilon|$.*

(ii) (Regularity) *If $h \in H^j(\Omega) \cap C^{j-1}(\bar{\Omega})$, $j \ge 1$, then $u \in H^{j+1}(\Omega) \cap C^j_0(\bar{\Omega})$.*

Theorem 5.12 is a corollary of a more general result, which holds for nonmonotone nonlinearities such as, for example,

$$f(t, x, u) = \begin{cases} \pm(\sin x)\, u^{2k} + h(t, x)\,, \\ \pm\, u^{2k} + u^{2k+1} + h(t, x)\,, \end{cases}$$

and also without any growth condition for f; see Theorems 1, 2 in [21], [22].

Remark 5.13. The hypothesis $h > 0$ enables us to prove the existence of a minimum of the reduced-action functional Φ.

Remark 5.14. The assumption $h \in N^{\perp}$ is not of a technical nature: if $h \notin N^{\perp}$, periodic solutions of (5.1) do not exist in any fixed ball $\{\|u\|_{L^\infty} \le R\}$ for ε small. Indeed, let $u := u_\varepsilon = v_\varepsilon + w_\varepsilon$, $v_\varepsilon \in V$, $w_\varepsilon \in W$, be a weak solution of (5.1) with $\|u_\varepsilon\|_{L^\infty} \le R$. Since w_ε satisfies the range equation $w_\varepsilon = \varepsilon\Box^{-1}\Pi_{N^\perp}(u_\varepsilon^{2k} + h)$, then $\|w_\varepsilon\|_{L^\infty} \le C(R)|\varepsilon|$. Moreover, $\Pi_N((v_\varepsilon + w_\varepsilon)^{2k} + h) = 0$, and since $\Pi_N v_\varepsilon^{2k} = 0$ by Lemma 2.18,

$$\Big\|\Pi_N h\Big\|_{L^2} = \Big\|\Pi_N((v_\varepsilon + w_\varepsilon)^{2k} - v_\varepsilon^{2k})\Big\|_{L^2} \to 0$$

as $\|w_\varepsilon\|_{L^\infty} \to 0$. Therefore $\Pi_N h = 0$ and $h \in N^{\perp}$.

Remark 5.15. **(Multiplicity)** For nonmonotone nonlinearities f, uniqueness of the solutions is NOT expected. For example, let f be as in (5.42) and $h(t,x) := -(v_0(t,x))^{2k}$ for some $v_0 \in V \setminus \{0\}$. By Lemma 2.18, $h \in N^{\perp}$. Equation (5.1) possesses, beyond the ε-small solution u of Theorem 5.12, also the other two (not small) solutions $\pm v_0$.

The difficulty in dealing with nonmonotone nonlinearities is well highlighted for nonlinearities f as in (5.42). In this case the variational inequality (5.23) vanishes identically for $\varepsilon = 0$, because

$$\int_\Omega \Big(\bar v^{2k} + h(t,x)\Big)\phi \not\equiv 0, \qquad \forall \phi \in V,$$

by Lemma 2.18 and $h \in N^{\perp}$.

Therefore, for deriving, if possible, the required a priori estimates, we have to develop the variational inequality (5.23) at higher orders in ε.

Perform in (5.1) with $f = u^{2k} + h$ the change of variables

$$u \to \varepsilon(H + u), \tag{5.43}$$

where H is a weak solution of

$$\begin{cases} H_{tt} - H_{xx} = h, \\ H(t,0) = H(t,\pi) = 0, \\ H(t + 2\pi, x) = H(t,x), \end{cases} \tag{5.44}$$

and since $h > 0$ in Ω, we can always choose

$$H(t,x) > 0, \quad \forall (t,x) \in \Omega,$$

by the "maximum principle" theorem proved in Section 5.4.4.

Therefore (renaming $\varepsilon^{2k} \to \varepsilon$) we look for solutions of

$$\begin{cases} u_{tt} - u_{xx} = \varepsilon f(t, x, u)\,, \\ u(t, 0) = u(t, \pi) = 0\,, \\ u(t + 2\pi, x) = u(t, x)\,, \end{cases}$$

where the nonlinear forcing term is now (still called f)

$$f(t, x, u) := (H + u)^{2k}\,.$$

Implementing a variational Lyapunov–Schmidt reduction as in Section 5.2, we obtain the existence of a constrained minimum $\bar{v} \in \bar{B}_R$, where the variational inequality

$$\int_\Omega \big(H + \bar{v} + w(\varepsilon, \bar{v})\big)^{2k} \phi \le 0 \tag{5.45}$$

is satisfied for any admissible variation $\phi \in V$.

As in Section 5.3, the required a priori estimate for the H^1-norm of $\bar{v}$ is proved in several steps by inserting into the variational inequality (5.45) suitable admissible variations. We shall derive first an L^{2k}-estimate for $\bar{v}$ (it is needed at least when $k \ge 2$), next, an L^∞ estimate, and, finally, the H^1 estimate.

We shall carry out the proof only for the more difficult case $k \ge 2$.

The following key "coercivity" estimate will be heavily exploited.

Lemma 5.16. (Coercivity estimate) *Let $B \in C(\bar{\Omega})$ with $B > 0$ in Ω. Then $\forall\, v \in N \cap L^{2k}(\Omega)$,*

$$\int_\Omega B v^{2k} \ge c_k(B) \int_\Omega v^{2k}\,, \tag{5.46}$$

where

$$c_k(B) := \frac{1}{4^k} \min_{\bar{\Omega}_{\alpha_k}} B > 0\,, \qquad \alpha_k := \frac{1}{4(1 + 2k)}\,, \tag{5.47}$$

and

$$\Omega_\alpha := \mathbf{T} \times (\alpha\pi, \pi - \alpha\pi) \subseteq \Omega\,.$$

The inequality (5.46) is not trivial because B can vanish at the boundary of $\partial\Omega$ (i.e., $B(t, 0) = B(t, \pi) = 0$). It holds because $v(t, x) = \hat{v}(t + x) - \hat{v}(t - x)$ is the linear superposition of two traveling waves that "spend a lot of time" far from the boundary $x = 0, x = \pi$.

Lemma 5.16 is a consequence of several lemmas.

Lemma 5.17. *For $p, q \in L^1(\mathbf{T})$,*

$$\int_{\Omega_\alpha} p(t + x) q(t - x)\, \mathrm{d}t\, \mathrm{d}x$$
$$= \frac{1}{2} \int_0^{2\pi} p(s)\, \mathrm{d}s \int_0^{2\pi} q(s)\, \mathrm{d}s - \frac{1}{2} \int_{-2\alpha\pi}^{2\alpha\pi} \int_0^{2\pi} p(y) q(z + y)\, \mathrm{d}y\, \mathrm{d}z \tag{5.48}$$

and

$$\int_{\Omega_\alpha} p(t+x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\Omega_\alpha} p(t-x)\,\mathrm{d}t\,\mathrm{d}x$$
$$= \pi(1-2\alpha)\int_0^{2\pi} p(s)\,\mathrm{d}s\,. \qquad (5.49)$$

For $f, g\colon \mathbf{R} \to \mathbf{R}$ continuous,

$$\int_{\Omega_\alpha} f(p(t+x))g(p(t-x))\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{\Omega_\alpha} f(p(t-x))g(p(t+x))\,\mathrm{d}t\,\mathrm{d}x\,, \qquad (5.50)$$

and $\forall a, b \in \mathbf{R}$,

$$(a-b)^{2k} \geq a^{2k} + b^{2k} - 2k(a^{2k-1}b + ab^{2k-1})\,. \qquad (5.51)$$

Proof. By the periodicity in t,

$$\int_{\Omega_\alpha} p(t+x)q(t-x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\widetilde{\Omega}_\alpha} p(t+x)q(t-x)\,\mathrm{d}t\,\mathrm{d}x\,,$$

where $\widetilde{\Omega}_\alpha := \{\alpha\pi < x < \pi(1-\alpha), -x < t < -x + 2\pi\}$. Under the change of variables $s_+ := t + x$, $s_- := t - x$, the domain $\widetilde{\Omega}_\alpha$ transforms into

$$\Big\{0 < s_+ < 2\pi,\ s_+ - 2\pi(1-\alpha) < s_- < s_+ - 2\pi\alpha\Big\}\,,$$

and therefore

$$\begin{aligned}
&\int_{\Omega_\alpha} p(t+x)q(t-x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_0^{2\pi} \mathrm{d}s_+\ p(s_+) \int_{-2\pi+s_++2\alpha\pi}^{s_+-2\alpha\pi} q(s_-)\,\mathrm{d}s_- \\
&\quad = \frac{1}{2}\int_0^{2\pi} \mathrm{d}s_+\ p(s_+) \left(\int_0^{2\pi} q(s)\,\mathrm{d}s - \int_{s_+-2\alpha\pi}^{s_++2\alpha\pi} q(s_-)\,\mathrm{d}s_-\right) \\
&\quad = \frac{1}{2}\int_0^{2\pi} p(s)\,\mathrm{d}s \int_0^{2\pi} q(s)\,\mathrm{d}s \\
&\qquad - \frac{1}{2}\int_0^{2\pi} \mathrm{d}s_+\ p(s_+) \int_{-2\alpha\pi}^{2\alpha\pi} q(s_+ + z)\,\mathrm{d}z\,,
\end{aligned}$$

and we obtain (5.48) by Tonelli's theorem (calling $s_+ = y$).

Formula (5.49) follows by (5.48) setting $q = 1$.

Since the change of variables $(t, x) \mapsto (t, \pi - x)$ leaves the domain Ω_α unchanged, then using the periodicity of p,

$$\int_{\Omega_\alpha} f(p(t+x))g(p(t-x))\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{\Omega_\alpha} f(p(t+\pi-x))g(p(t-\pi+x))\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi} f(p(t+\pi-x))g(p(t+\pi+x))\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi} f(p(t-x))g(p(t+x))\,\mathrm{d}t\,\mathrm{d}x\,,$$

proving (5.50). For the elementary inequality (5.51) we refer to [22]. ■

Lemma 5.18. *Let* $v \in N \cap L^{2k}(\Omega)$. *Then*

$$\int_\Omega v^{2k} \le \pi 4^k \int_0^{2\pi} \hat{v}^{2k} \tag{5.52}$$

and

$$\int_{\Omega_\alpha} v^{2k} \ge 2\pi\Big[1-2(1+2k)\alpha\Big]\int_0^{2\pi} \hat{v}^{2k} \ge \pi\int_0^{2\pi} \hat{v}^{2k} \tag{5.53}$$

if $0 \le \alpha \le 1/4(1+2k)$.

Proof. The convexity inequality

$$(a-b)^{2k} \le 2^{2k-1}(a^{2k}+b^{2k})\,, \quad \forall\, a,b \in \mathbf{R}\,,$$

yields

$$\int_\Omega v^{2k} = \int_\Omega (v_+ - v_-)^{2k} \le 2^{2k-1}\int_\Omega \hat{v}^{2k}(t+x) + \hat{v}^{2k}(t-x)\,,$$

which, using Lemma 2.20, proves (5.52).

Using (5.51), (5.49), and (5.50),

$$\int_{\Omega_\alpha} v^{2k} = \int_{\Omega_\alpha} (v_+ - v_-)^{2k} \overset{(5.51)}{\ge} \int_{\Omega_\alpha} v_+^{2k} + v_-^{2k} - 2k\int_{\Omega_\alpha} v_+^{2k-1}v_- + v_+v_-^{2k-1}$$
$$= 2\pi(1-2\alpha)\int_0^{2\pi} \hat{v}^{2k} - 4k\int_{\Omega_\alpha} v_+^{2k-1}v_-\,. \tag{5.54}$$

By (5.48) and since $\hat{v}$ has zero average,

$$\left|\int_{\Omega_\alpha} v_+^{2k-1}v_-\right| = \frac{1}{2}\left|\int_{-2\pi\alpha}^{2\pi\alpha}\int_0^{2\pi} \hat{v}^{2k-1}(y)\hat{v}(y+z)\,\mathrm{d}y\,\mathrm{d}z\right|$$
$$\le \frac{1}{2}\int_{-2\pi\alpha}^{2\pi\alpha}\int_0^{2\pi} |\hat{v}(y)|^{2k-1}|\hat{v}(y+z)|\,\mathrm{d}y\,\mathrm{d}z\,. \tag{5.55}$$

By Hölder's inequality with $p := 2k/(2k-1)$ and $q := 2k$,

$$\int_0^{2\pi} |\hat{v}(y)|^{2k-1}|\hat{v}(y+z)|\,\mathrm{d}y \le \left(\int_0^{2\pi} |\hat{v}(y)|^{2k}\,\mathrm{d}y\right)^{\frac{2k-1}{2k}} \left(\int_0^{2\pi} |\hat{v}(y+z)|^{2k}\,\mathrm{d}y\right)^{\frac{1}{2k}}$$
$$= \int_0^{2\pi} |\hat{v}(y)|^{2k}\,\mathrm{d}y\,,$$

where in the equality we have used the periodicity of $\hat{v}$. Inserting the last inequality in (5.55) yields

$$\left|\int_{\Omega_\alpha} v_+^{2k-1} v_-\right| \le 2\pi\,\alpha \int_0^{2\pi} |\hat{v}(y)|^{2k}\,\mathrm{d}y\,. \tag{5.56}$$

Inserting (5.56) in (5.54) gives us (5.53). ■

Proof of Lemma 5.16. Since $B > 0$ in Ω and using Lemma 5.18,

$$\int_\Omega B v^{2k} \ge \min_{\bar{\Omega}_{\alpha_k}} B \int_{\Omega_{\alpha_k}} v^{2k} \overset{(5.53)}{\ge} \pi \min_{\bar{\Omega}_{\alpha_k}} B \int_0^{2\pi} \hat{v}^{2k} \overset{(5.52)}{\ge} \frac{1}{4^k} \min_{\bar{\Omega}_{\alpha_k}} B \int_\Omega v^{2k}\,.$$

5.4.1 Step 1: the L^{2k} Estimate

Insert $\phi := \bar{v}$ in the variational inequality (5.45). Note that ϕ is an admissible variation, since

$$\langle \bar{v}, \phi\rangle_{H^1} = \|\bar{v}\|_{H^1}^2 > 0\,.$$

Using the variational inequality (5.45), setting $\bar{w} := w(\varepsilon, \bar{v})$,

$$\begin{aligned}\int_\Omega (\bar{v}+H)^{2k}\bar{v} &= \int_\Omega (\bar{v}+H+\bar{w})^{2k}\bar{v} + \int_\Omega \left[(\bar{v}+H)^{2k} - (\bar{v}+\bar{w}+H)^{2k}\right]\bar{v} \\ &\overset{(5.45)}{\le} \int_\Omega \left[(\bar{v}+H)^{2k} - (\bar{v}+\bar{w}+H)^{2k}\right]\bar{v} \\ &\overset{(5.19)}{\le} |\varepsilon| C_1(R)\|\bar{v}\|_{L^1} \le 1 \end{aligned} \tag{5.57}$$

for $|\varepsilon| \le \varepsilon_1(R)$, where $0 < \varepsilon_1(\cdot) \le \varepsilon_0(\cdot)$.

Since $\int_\Omega \bar{v}^{2k+1} = 0$ by (2.48), and using Lemma 5.16,

$$\begin{aligned} 1 \overset{(5.57)}{\ge} \int_\Omega (\bar{v}+H)^{2k}\bar{v} &= \int_\Omega \left[(\bar{v}+H)^{2k} - \bar{v}^{2k}\right]\bar{v} \\ &= \int_\Omega 2kH\bar{v}^{2k} + \sum_{j=0}^{2k-2}\binom{2k}{j}\bar{v}^{j+1}H^{2k-j} \\ &\overset{(5.46)}{\ge} 2kc_k(H)\|\bar{v}\|_{L^{2k}}^{2k} - \kappa_1\|\bar{v}\|_{L^{2k}}^{2k-1} - \kappa_2\|\bar{v}\|_{L^{2k}}\,, \end{aligned} \tag{5.58}$$

where $c_k(H) > 0$ is defined in (5.47) (recall that $H > 0$), and we have used the Hölder inequality to estimate $\|\bar{v}\|_{L^i} \le C_{i,k}\|\bar{v}\|_{L^{2k}}$ ($i \le 2k-1$).

Then (5.58) implies

$$\|\bar{v}\|_{L^{2k}} \le \kappa_3 \qquad \text{for} \quad |\varepsilon| \le \varepsilon_1(R)\,. \tag{5.59}$$

5.4.2 Step 2: the L^∞ Estimate

Let ϕ be the same admissible variation as in Section 5.3.1.

Using the variational inequality (5.45), setting $\bar{w} := w(\varepsilon, \bar{v})$,

$$\begin{aligned}\int_\Omega (\bar{v}+H)^{2k}\phi &= \int_\Omega (\bar{v}+H+\bar{w})^{2k}\phi + \int_\Omega \Big[(\bar{v}+H)^{2k} - (\bar{v}+\bar{w}+H)^{2k}\Big]\phi \\ &\overset{(5.45)}{\leq} \int_\Omega \Big[(\bar{v}+H)^{2k} - (\bar{v}+\bar{w}+H)^{2k}\Big]\phi \\ &\overset{(5.19)}{\leq} |\varepsilon| C_2(R)\|\phi\|_{L^1} \leq \|\phi\|_{L^1} \end{aligned} \tag{5.60}$$

for $|\varepsilon| \leq \varepsilon_2(R)$, where $0 < \varepsilon_2(\cdot) \leq \varepsilon_1(\cdot)$.

Now, using that $\int_\Omega \bar{v}^{2k}\phi = 0$ by Lemma 2.18,

$$\begin{aligned}\|\phi\|_{L^1} &\overset{(5.60)}{\geq} \int_\Omega (\bar{v}+H)^{2k}\phi = \int_\Omega \Big[(\bar{v}+H)^{2k} - \bar{v}^{2k}\Big]\phi \\ &\geq \int_\Omega 2kH\bar{v}^{2k-1}\phi - \kappa_4\big(\|\bar{v}\|_{L^\infty}^{2k-2}+1\big)\|\phi\|_{L^1},\end{aligned}$$

which implies, since

$$\int_\Omega H\bar{v}^{2k-1}\phi \geq \kappa_5 \int_{\Omega_{1/4}} \phi\bar{v}^{2k-1}$$

(because $\bar{v}\phi \geq 0$ and $\min_{\Omega_{1/4}} H > 0$), that

$$\begin{aligned}\int_{\Omega_{1/4}} \bar{v}^{2k-1}\phi &\leq \kappa_6\big(\|\bar{v}\|_{L^\infty(\Omega)}^{2k-2}+1\big)\|\phi\|_{L^1(\Omega)} \\ &\overset{(5.32)}{\leq} \kappa_7\big(\|\bar{v}\|_{L^\infty(\Omega)}^{2k-2}+1\big)\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})}.\end{aligned} \tag{5.61}$$

We have to give a lower bound for the positive integral

$$\int_{\Omega_{1/4}} \bar{v}^{2k-1}\phi = \int_{\Omega_{1/4}} (\bar{v}\phi)\bar{v}^{2k-2} = \int_{\Omega_{1/4}} (\bar{v}\phi)(\bar{v}_+ - \bar{v}_-)^{2k-2}.$$

Using (5.51),

$$\begin{aligned}\int_{\Omega_{1/4}} \bar{v}^{2k-1}\phi &\geq \int_{\Omega_{1/4}} \bar{v}\phi\Big[\bar{v}_+^{2k-2} + \bar{v}_-^{2k-2} - (2k-2)(\bar{v}_+^{2k-3}\bar{v}_- + \bar{v}_+\bar{v}_-^{2k-3})\Big] \\ &= 2\int_{\Omega_{1/4}} \bar{v}_+^{2k-1}q_+ - \bar{v}_+^{2k-1}q_- + \bar{v}_+^{2k-2}\bar{v}_-q_- - \bar{v}_+^{2k-2}\bar{v}_-q_+ \\ &\quad + (2k-2)\Big[-\bar{v}_+^{2k-2}\bar{v}_-q_+ + \bar{v}_+^{2k-2}\bar{v}_-q_- \\ &\qquad - \bar{v}_+^{2k-3}\bar{v}_-^2q_- + \bar{v}_+^{2k-3}\bar{v}_-^2q_+\Big]\end{aligned}$$

$$\geq 2\int_{\Omega_{1/4}} \bar{v}_+^{2k-1} q_+ \tag{5.62}$$

$$-2\int_{\Omega_{1/4}} \bar{v}_+^{2k-1} q_- + (2k-1)\bar{v}_+^{2k-2}\bar{v}_- q_+ \tag{5.63}$$

$$+ (2k-2)\bar{v}_+^{2k-3}\bar{v}_-^2 q_- \,, \tag{5.64}$$

where in the equality we have used (5.50) and in the last inequality the fact that $\bar{v}_+ q_+$, $\bar{v}_- q_- \geq 0$ (since $\lambda q(\lambda) \geq 0$), and so $\bar{v}_+^{2k-2}\bar{v}_- q_-$, $\bar{v}_+^{2k-3}\bar{v}_-^2 q_+ \geq 0$.

The dominant term (5.62) is estimated by

$$2\int_{\Omega_{1/4}} \bar{v}_+^{2k-1} q_+ \overset{(5.49)}{=} 2\pi(1-\frac{2}{4})\int_0^{2\pi} \hat{\bar{v}}^{2k-1}(s) q(\hat{\bar{v}}(s))\mathrm{d}s$$

$$\geq \pi M^{2k-1}\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \tag{5.65}$$

because $\lambda^{2k-1} q(\lambda) \geq M^{2k-1}|q(\lambda)|$.

The three terms in (5.63)–(5.64) are estimated by

$$\left|2\int_{\Omega_{1/4}} \bar{v}_+^{2k-1} q_-\right| \leq 2\int_{\Omega} \left|\bar{v}_+^{2k-1}\right| |q_-| \leq \|\hat{\bar{v}}\|_{L^{2k-1}(\mathbf{T})}^{2k-1} \|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \,,$$

$$\left|2\int_{\Omega_{1/4}} \left(\bar{v}_+^{2k-2} q_+\right)\bar{v}_-\right| \leq \|\hat{\bar{v}}^{2k-2} q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \|\hat{\bar{v}}\|_{L^1(\mathbf{T})}$$

$$\leq (2M)^{2k-2}\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \|\hat{\bar{v}}\|_{L^1(\mathbf{T})} \,,$$

and

$$\left|2\int_{\Omega_{1/4}} \bar{v}_+^{2k-3}\left(\bar{v}_-^2 q_-\right)\right| \leq \|\hat{\bar{v}}^{2k-3}\|_{L^1(\mathbf{T})} \|\hat{\bar{v}}^2 q(\hat{\bar{v}})\|_{L^1(\mathbf{T})}$$

$$\leq (2M)^2 \|\hat{\bar{v}}\|_{L^{2k-3}(\mathbf{T})}^{2k-3} \|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \,.$$

By the previous inequalities, (5.65), Hölder inequality,[3] and using the L^{2k}-estimate (5.59) for $\bar{v}$,

$$\int_{\Omega_{1/4}} \bar{v}^{2k-1}\phi \geq \pi M^{2k-1}\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} - \kappa_7(M^{2k-2}+1)\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \,. \tag{5.66}$$

By (5.61) and (5.66),

$$M^{2k-1}\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \leq \kappa_8\left(\|\bar{v}\|_{L^\infty(\Omega)}^{2k-2} + M^{2k-2} + 1\right)\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})}$$

$$\leq \kappa_9\left(M^{2k-2}+1\right)\|q(\hat{\bar{v}})\|_{L^1(\mathbf{T})} \,,$$

and finally $M^{2k-1} \leq \kappa_9(M^{2k-2}+1)$. By our choice of $M := \frac{1}{2}\|\hat{\bar{v}}\|_{L^\infty(\mathbf{T})}$ (see (5.27)) the L^∞-estimate follows:

$$\|\bar{v}\|_{L^\infty} \leq \kappa_{10} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_2(R) \,. \tag{5.67}$$

---

[3] To estimate $\|\hat{\bar{v}}\|_{L^j(\mathbf{T})} \leq C_{j,k}\|\hat{\bar{v}}\|_{L^{2k}(\mathbf{T})}$ for $j < 2k$.

5.4.3 Step 3: The H^1 Estimate

Inserting the admissible variation $\phi = -D_{-h}D_h\bar{v}$ in the variational inequality (5.45) yields

$$\|\phi\|_{H^1} \overset{(5.45)}{\geq} \int_\Omega \left(H+\bar{v}+\bar{w}\right)^{2k}\phi$$
$$\overset{(5.35)}{=} \int_\Omega D_h(H+\bar{v}+\bar{w})^{2k} D_h\bar{v} \overset{(5.37)}{\to} \int_\Omega \left[(H+\bar{v}+\bar{w})^{2k}\right]_t \bar{v}_t \quad (5.68)$$

as $h \to 0$ since $\partial_t\left[(H+\bar{v}+\bar{w})^{2k}\right] \in L^2$.

Since $\|\bar{w}\|_E \leq C_0(R)|\varepsilon|$, by (5.67) and the Cauchy–Schwarz inequality,

$$\int_\Omega \left[\left(H+\bar{v}+\bar{w}\right)^{2k}\right]_t \bar{v}_t = 2k\int_\Omega \left(H+\bar{v}+\bar{w}\right)^{2k-1}\left(H_t+\bar{v}_t+\bar{w}_t\right)\bar{v}_t$$
$$\geq 2k\int_\Omega (\bar{v}+H)^{2k-1}\bar{v}_t^2 - o(1)\|\bar{v}_t\|_{L^2}^2 - \kappa_{11}\|\bar{v}_t\|_{L^2},$$

and by (5.68),

$$\int_\Omega (\bar{v}+H)^{2k-1}\bar{v}_t^2 \leq \kappa_{12}\|\bar{v}_t\|_{L^2} + o(1)\|\bar{v}_t\|_{L^2}^2. \quad (5.69)$$

Since $\bar{v}, \bar{v}_t \in N$, then $\int_\Omega \bar{v}^{2k-1}\bar{v}_t^2 = 0$ by Lemma 2.18 and using the elementary inequality[4]

$$(a+b)^{2k-1} - a^{2k-1} \geq 4^{1-k}\, b^{2k-1}, \qquad \forall\, a \in \mathbf{R},\ b > 0,$$

gives

$$\int_\Omega (\bar{v}+H)^{2k-1}\bar{v}_t^2 = \int_\Omega \left[(\bar{v}+H)^{2k-1} - \bar{v}^{2k-1}\right]\bar{v}_t^2$$
$$\geq 4^{1-k}\int_\Omega H^{2k-1}\bar{v}_t^2 \geq 4^{1-k}c_1(H^{2k-1})\int_\Omega \bar{v}_t^2, \quad (5.70)$$

where $c_1(\cdot)$ was defined in (5.47). By (5.69) and (5.70),

$$\|\bar{v}_t\|_{L^2}^2 \leq \kappa_{13}\|\bar{v}_t\|_{L^2} + o(1)\|\bar{v}_t\|_{L^2}^2,$$

and we finally deduce that there exists $0 < \varepsilon_3(R) \leq \varepsilon_2(R)$ such that

$$\|\bar{v}\|_{H^1} < \kappa_{14} \qquad \forall\, |\varepsilon| \leq \varepsilon_3(R). \qquad \blacksquare$$

[4] Dividing by b^{2k-1} and setting $x := a/b$, we have to prove $f(x) := (x+1)^{2k-1} - x^{2k-1} \geq 4^{1-k}$ $\forall x \in \mathbf{R}$. An elementary calculation shows that $x = -1/2$ is the unique minimum point of $f(x)$ and $f(-1/2) = 4^{1-k}$.

Proof of Theorem 5.12 *completed.* Take

$$R_* := \kappa_{14} \quad \text{and} \quad \varepsilon_* := \varepsilon_3(R_*)\,.$$

Therefore

$$\|\bar{v}(\varepsilon)\|_{H^1} < R_* \qquad \forall\, |\varepsilon| \le \varepsilon_*$$

is an interior minimum of Φ in $B_{R_*} := \{\|v\|_{H^1} < R_*\}$. Recalling the change of variables (5.43),

$$u := \varepsilon\Big(H + \bar{v}(\varepsilon^{2k}) + w(\varepsilon^{2k}, \bar{v}(\varepsilon^{2k}))\Big) \in E$$

is a weak solution of (5.1) with $f = u^{2k} + h$. ■

We shall not prove the higher regularity of the solution. In view of the regularity result of [43]—which holds just for strictly monotone nonlinearities—it is somewhat surprising that the weak solution u of Theorem 5.12 is actually smooth.

5.4.4 The "Maximum Principle"

Theorem 5.19. ([22]) *Let* $h \in N^{\perp}$, $h > 0$ *a.e. in* Ω. *Then there exists a weak solution* $H \in E$ *of (5.44) satisfying* $H > 0$. *In particular,*

$$H(t,x) := \frac{1}{2}\int_0^{\kappa}\int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi - \frac{1}{2}\int_{\kappa}^{x}\int_{t-x+\xi}^{t+x-\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \tag{5.71}$$

for a suitable $\kappa \in (0,\pi)$.

Proof. **Step 1:** $H(t,x) \in H^1(\Omega) \cap C^{1/2}(\bar{\Omega})$ *for any* $\kappa \in (0,\pi)$ *and*

$$\begin{aligned}
2(\partial_t H) = &\int_0^{\kappa} \Big(h(t-x+\xi,\xi) - h(t-x-\xi,\xi)\Big)\,\mathrm{d}\xi \\
&- \int_{\kappa}^{x} \Big(h(t+x-\xi,\xi) - h(t-x+\xi,\xi)\Big)\,\mathrm{d}\xi \in L^2(\Omega)\,, \qquad (5.72)\\
2(\partial_x H) = &\int_0^{\kappa} \Big(-h(t-x+\xi,\xi) + h(t-x-\xi,\xi)\Big)\,\mathrm{d}\xi \\
&- \int_{\kappa}^{x} \Big(h(t+x-\xi,\xi) + h(t-x+\xi,\xi)\Big)\,\mathrm{d}\xi \in L^2(\Omega)\,. \qquad (5.73)
\end{aligned}$$

We shall prove that the first integral on the right-hand side of (5.71),

$$H_1(t,x) := \frac{1}{2}\int_0^{\kappa}\int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\,,$$

belongs to $C^{1/2}(\bar{\Omega}) \cap H^1(\Omega)$, the second integral being analogous. Defining

$$T(t,x) := \Big\{(\tau,\xi)\in\Omega \ \Big|\ t-x-\xi<\tau<t-x+\xi\,,\ \ 0<\xi<\kappa\Big\}\,,$$

we can write $H_1(t,x) := \frac{1}{2}\int_{T(t,x)} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi$. Since $|T(t,x)| = \kappa^2 \le \pi^2$,

$$\big|H_1(t,x)\big| \le \frac{1}{2}\int_\Omega \mathbf{1}_{T(t,x)}(\tau,\xi)|h(\tau,\xi)|\,\mathrm{d}\tau\,\mathrm{d}\xi \le \frac{\pi}{2}\|h\|_{L^2(\Omega)}\,,$$

using the Cauchy–Schwarz inequality.

For $i = 1, 2$ and $(t_i, x_i) \in \Omega$, let us define $T_i := T(t_i, x_i)$. We then obtain

$$|T_1\setminus T_2| = |T_2\setminus T_1| \le \pi\big(|t_1-t_2|+|x_1-x_2|\big)\,,$$

and using again the Cauchy–Schwarz inequality,

$$\begin{aligned}|H_1(t_1,x_1)-H_1(t_2,x_2)| &\le \frac{1}{2}\int_{T_1\setminus T_2}|h| + \frac{1}{2}\int_{T_2\setminus T_1}|h| \\ &\le \sqrt{\pi}\big(|t_1-t_2|+|x_1-x_2|\big)^{1/2}\|h\|_{L^2(\Omega)}\end{aligned}$$

and $H_1 \in C^{1/2}(\bar\Omega)$.

We now prove that $H_1 \in H^1(\Omega)$ and that $\partial_t H_1 = -\partial_x H_1 = f_1$, where

$$f_1(t,x) := \frac{1}{2}\int_0^\kappa \Big(h(t-x+\xi,\xi) - h(t-x-\xi,\xi)\Big)\,\mathrm{d}\xi \in L^2(\Omega)\,.$$

We first justify that $f_1 \in L^2(\Omega)$. Since

$$\begin{aligned}&|h(t-x+\xi,\xi)|^2 + |h(t-x-\xi,\xi)|^2 \\ &\qquad\ge \frac{1}{2}\Big|h(t-x+\xi,\xi) - h(t-x-\xi,\xi)\Big|^2\,,\end{aligned}$$

by periodicity with respect to t we obtain that $\forall\, x \in (0,\pi)$,

$$\begin{aligned}\|h\|^2_{L^2(\Omega)} &\ge \int_0^\kappa\int_0^{2\pi}|h(t,\xi)|^2\mathrm{d}t\,\mathrm{d}\xi \\ &\ge \frac{1}{4}\int_0^\kappa\int_0^{2\pi}|h(t-x+\xi,\xi)-h(t-x-\xi,\xi)|^2\mathrm{d}t\,\mathrm{d}\xi\,.\end{aligned}$$

Integrating the previous inequality in the variable x between 0 and π, applying Fubini's theorem and the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned}\pi\,\|h\|^2_{L^2(\Omega)} &\ge \frac{1}{4}\int_0^\pi\int_0^\kappa\int_0^{2\pi}|h(t-x+\xi,\xi)-h(t-x-\xi,\xi)|^2\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega\int_0^\kappa|h(t-x+\xi,\xi)-h(t-x-\xi,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ &\ge \frac{1}{4\kappa}\int_\Omega\bigg(\int_0^\kappa|h(t-x+\xi,\xi)-h(t-x-\xi,\xi)|\,\mathrm{d}\xi\bigg)^2\,\mathrm{d}t\,\mathrm{d}x\,. \\ &\ge \frac{1}{\kappa}\int_\Omega f_1^2(t,x)\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{\kappa}\|f_1\|^2_{L^2}\,.\end{aligned}$$

Finally, we prove that $\partial_x H_1 = -f_1$, with $\partial_t H_1 = f_1$ being analogous. By Fubini's theorem,

$$
\begin{aligned}
2\int_\Omega H_1\phi_x &= \int_0^{2\pi}\int_0^{\kappa}\int_0^{\pi}\int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\phi_x(t,x)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t\\
&= \int_0^{2\pi}\int_0^{\kappa}\bigg(\int_{t+\xi-\pi}^{t+\xi} h\int_0^{t+\xi-\tau}\phi_x\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t-\xi}^{t+\xi-\pi} h\int_0^{\pi}\phi_x\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad + \int_{t-\xi-\pi}^{t-\xi} h\int_{t-\xi-\tau}^{\pi}\phi_x\,\mathrm{d}x\,\mathrm{d}\tau\bigg)\,\mathrm{d}\xi\,\mathrm{d}t\\
&= \int_0^{2\pi}\int_0^{\kappa}\bigg(\int_{t+\xi-\pi}^{t+\xi} h(\tau,\xi)\phi(t,t+\xi-\tau)\,\mathrm{d}\tau\\
&\qquad - \int_{t-\xi-\pi}^{t-\xi} h(\tau,\xi)\phi(t,t-\xi-\tau)\,\mathrm{d}\tau\bigg)\,\mathrm{d}\xi\,\mathrm{d}t\\
&= \int_0^{2\pi}\int_0^{\kappa}\bigg(-\int_\pi^0 h(t+\xi-x,\xi)\phi(t,x)\,\mathrm{d}x\\
&\qquad + \int_\pi^0 h(t-\xi-x,\xi)\phi(t,x)\,\mathrm{d}x\bigg)\,\mathrm{d}\xi\,\mathrm{d}t = 2\int_\Omega f_1\phi\,.
\end{aligned}
$$

With analogous computations for the second addend of H in (5.71) we derive (5.72) and (5.73).

Step 2: *There exists $\kappa \in (0,\pi)$ such that $H(t,x)$ satisfies the Dirichlet boundary conditions $H(t,0) = H(t,\pi) = 0\ \forall\, t \in \mathbf{T}$.*

By (5.71), the function H satisfies, for any $\kappa \in (0,\pi)$, $H(t,0) = 0\ \forall\, t \in \mathbf{T}$. It remains to find κ, imposing $H(t,\pi) = 0$. Taking $x = \pi$ in (5.71) we obtain

$$
\begin{aligned}
H(t,\pi) :=\ & \frac{1}{2}\int_0^{\kappa}\int_{t-\pi-\xi}^{t-\pi+\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi - \frac{1}{2}\int_\kappa^{\pi}\int_{t-\pi+\xi}^{t+\pi-\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\\
=\ & \frac{1}{2}\int_0^{\pi}\int_{t-\pi-\xi}^{t-\pi+\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi - \frac{1}{2}\int_\kappa^{\pi}\int_{t-\pi-\xi}^{t+\pi-\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\\
=\ & \frac{1}{2}\int_0^{\pi}\int_{-\xi}^{\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi - \frac{1}{2}\int_\kappa^{\pi}\int_{-\pi}^{\pi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\\
=:\ & c - \chi(\kappa)\,,
\end{aligned}
$$

where in the last line we have used the periodicity of $h(\cdot,\xi)$. Since $\chi(\kappa)$ is a continuous function, $\chi(0) > c > 0$ and $\chi(\pi) = 0$, there exists $\kappa \in (0,\pi)$ solving $\chi(\kappa) = c$.

Step 3: *$H \in E$ is a weak solution of (5.44), namely*

$$
\int_\Omega \phi_t H_t - \phi_x H_x + \phi h = 0\,, \qquad \forall\,\phi \in C_0^1(\bar{\Omega})\,. \tag{5.74}
$$

By Fubini's theorem and periodicity we get

$$\int_\Omega \left(\phi_t(t,x)\int_0^\kappa h(t-x+\xi,\xi)\mathrm{d}\xi + \phi_x(t,x)\int_0^\kappa h(t-x+\xi,\xi)\mathrm{d}\xi\right)\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^\pi\int_0^\kappa\int_0^{2\pi}\Big(\phi_t(t,x)h(t-x+\xi,\xi)+\phi_x(t,x)h(t-x+\xi,\xi)\Big)\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x$$
$$= \int_0^\pi\int_0^\kappa\int_0^{2\pi}\Big(\phi_t(t+x-\xi,x)+\phi_x(t+x-\xi,x)\Big)h(t,\xi)\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x$$
$$= \int_0^\kappa\int_0^{2\pi} h(t,\xi)\int_0^\pi \frac{d}{dx}\Big(\phi(t+x-\xi,x)\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi = 0 \tag{5.75}$$

by Dirichlet boundary conditions. Analogously,

$$-\int_\Omega \left(\phi_t(t,x)\int_0^\kappa h(t-x-\xi,\xi)\mathrm{d}\xi + \phi_x(t,x)\int_0^\kappa h(t-x-\xi,\xi)\mathrm{d}\xi\right)\mathrm{d}t\,\mathrm{d}x = 0\,. \tag{5.76}$$

Moreover, again by Fubini's theorem,

$$\int_\Omega \left(-\phi_t(t,x)\int_\kappa^x h(t+x-\xi,\xi)\mathrm{d}\xi + \phi_x(t,x)\int_\kappa^x h(t+x-\xi,\xi)\mathrm{d}\xi\right)\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^\pi\int_\kappa^x\int_0^{2\pi}\Big(-\phi_t(t,x)h(t+x-\xi,\xi)+\phi_x(t,x)h(t+x-\xi,\xi)\Big)\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x$$
$$= \int_0^\pi\int_\kappa^x\int_0^{2\pi}\Big(-\phi_t(t-x+\xi,x)+\phi_x(t-x+\xi,x)\Big)h(t,\xi)\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x$$
$$= \int_0^{2\pi}\int_0^\pi h(t,\xi)\int_\xi^\pi \frac{\mathrm{d}}{\mathrm{d}x}\Big(\phi(t-x+\xi,x)\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi$$
$$-\int_0^{2\pi}\int_0^\kappa h(t,\xi)\int_0^\pi \frac{\mathrm{d}}{\mathrm{d}x}\Big(\phi(t-x+\xi,x)\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi = -\int_\Omega h\phi\,, \tag{5.77}$$

and analogously,

$$\int_\Omega \left(\phi_t(t,x)\int_\kappa^x h(t-x+\xi,\xi)\mathrm{d}\xi + \phi_x(t,x)\int_\kappa^x h(t-x+\xi,\xi)\mathrm{d}\xi\right)\mathrm{d}t\,\mathrm{d}x = -\int_\Omega h\phi\,. \tag{5.78}$$

Summing (5.75), (5.76), (5.77), (5.78) and recalling (5.72), (5.73), we get (5.74).

Step 4: $H(t,x) > 0$ *in* Ω.

First case: $0 < x \le \kappa$. By (5.71) and geometrical considerations on the domains of the integrals, we derive that for $0 < x < \kappa$, $H(t,x) = \int_\Theta h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi$, where $\Theta := \Theta_{t,x}$ is the trapezoidal region in Ω with a vertex in $(\tau,\xi) = (t,x)$ and delimited by the straight lines $\tau = t-x+\xi$, $\tau = t+x-\xi$, $\xi = \kappa$, and $\tau = t-x-\xi$. Since $h > 0$ a.e. in Ω, we conclude that $H(t,x) > 0$.

Second case: $\kappa < x < \pi$. Since $H(t+\pi-x,\pi) = 0$, we have, by (5.71),

$$\int_0^\kappa\int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi = \int_\kappa^\pi\int_{t-x+\xi}^{t-x-\xi+2\pi} h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\,.$$

Therefore, substituting in (5.71), we get, for $\kappa < x < \pi$, the expression $H(t,x) = \int_\Theta h(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi$ where now $\Theta := \Theta_{t,x}$ is the trapezoidal region in Ω with a vertex in $(\tau,\xi) = (t,x)$ and delimited by the straight lines $\tau = t - x + \xi$, $\tau = t - x - \xi + 2\pi$, $\xi = \kappa$, and $\tau = t + x - \xi$. Since $h > 0$ a.e. in Ω, we conclude also in this case that $H(t,x) > 0$. ■

Appendix A

Hamiltonian PDEs

We briefly present the Hamiltonian formulation of the nonlinear Schrödinger equation, the beam equation, the KdV equation, and the Euler equations of hydrodynamics. We refer to [82] for a complete presentation of the symplectic structures of Hamiltonian PDEs.

A.1 The Nonlinear Schrödinger Equation

$$\text{(NLS)} \qquad \begin{cases} \mathrm{i}\, u_t - \Delta u + f(x, |u|^2)u = 0\,, \\ u(x + 2\pi k) = u(x) \qquad \forall k \in \mathbf{Z}^d\,,\ \forall x \in \mathbf{R}^d\,, \end{cases}$$

with periodic boundary conditions (similarly for Dirichlet or Neumann boundary conditions) can be written in complex Hamiltonian form

$$\partial_t u = \mathrm{i}\, \partial_{\bar{u}} H$$

with Hamiltonian

$$H := \int_{\mathbf{T}^d} |\nabla u|^2 + F(x, |u|^2)\, \mathrm{d}x\,,$$

where $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$, $(\partial_s F)(x, s) = f(x, s)$ and $\partial_{\bar{u}} H = -\Delta u + f(x, |u|^2)u$ is the gradient with respect to the complex L^2-scalar product.

A.2 The Beam Equation

$$\begin{cases} u_{tt} + u_{xxxx} + f(x, u) = 0\,, \\ u(t, 0) = u(t, \pi) = 0\,, \\ u_{xx}(t, 0) = u_{xx}(t, \pi) = 0\,, \end{cases}$$

with Dirichlet and hinged boundary conditions, is, like the wave equation, a second-order Lagrangian equation with Lagrangian

$$\mathcal{L}(u, u_t) := \int_0^{\pi} \frac{u_t^2}{2} - \frac{u_{xx}^2}{2} - F(x, u)\,\mathrm{d}x\,,$$

where $\partial_u F(x, u) = f(x, u)$. The Hamiltonian of the beam equation is

$$H(u, p) := \int_0^{\pi} \frac{p^2}{2} + \frac{u_{xx}^2}{2} + F(x, u)\,\mathrm{d}x$$

with symplectic structure

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

A.3 The KdV Equation

$$\begin{cases} u_t - uu_x + u_{xxx} = 0\,, \\ u(x + 2\pi) = u(x)\,, \end{cases}$$

resp. with $u(x) \to 0$ as $|x| \to +\infty$, can be written in the Hamiltonian form (0.1) with Hamiltonian

$$H(u) := \int_{\mathbf{T}} \left(\frac{u_x^2}{2} + \frac{u^3}{6} \right) \mathrm{d}x$$

defined on the phase space $H^1(\mathbf{T})$ of functions u periodic in x, resp. on the space $H^1(\mathbf{R})$ of functions vanishing at infinity (solitons).

The antisymmetric nondegenerate[1] operator is

$$J := \partial_x\,,$$

and $\nabla H(u) = -u_{xx} + (u^2/2)$ is the gradient with respect to the L^2-scalar product.

Other Hamiltonian structures are possible to express the KdV equation; see, for example, [85] and [76].

A.4 The Euler Equations of Hydrodynamics

The Hamiltonian formulation of the "gravity-capillarity water waves" problem describing the evolution of a perfect, incompressible, irrotational fluid under the action of gravity and surface tension is more complex, and it is due to Zakharov [134].

The unknowns of the problem are the free surface $y = \eta(x)$, which bounds the variable fluid domain

[1] ∂_x is only weakly nondegenerate since it vanishes on constants.

$$S_\eta := \left\{(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid -h \le y \le \eta(x)\right\}$$

with $n = 2, 3$, and the velocity potential $\Phi\colon S_\eta \to \mathbf{R}$, i.e., the irrotational velocity field $u = \nabla\Phi$.

The gravity-capillarity water-waves problem, see, e.g., [125], can be written as

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = \sigma\tau & \text{at } y = \eta(x) \quad \text{(Bernoulli condition)}\,, \\ \Delta\Phi = 0 & \text{in } S_\eta \quad \text{(incompressibility)}\,, \\ \partial_y \Phi = 0 & \text{at } y = -h \quad \text{(impermeability)}\,, \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } y = \eta(x) \quad \text{(kinematic condition)}\,, \end{cases} \tag{A.1}$$

where g is the acceleration of gravity, σ is the surface tension, and

$$\tau := \operatorname{div} \frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}}$$

is the mean curvature of the free surface (we suppose the density of the fluid to be $\rho = 1$). Further boundary conditions are

$$\eta\,, \nabla\Phi \to 0 \quad \text{for} \quad |x| \to +\infty$$

(solitons) or periodic boundary conditions

$$\eta(x + \Lambda) = \eta(x)\,, \;\; \Phi(x + \Lambda, y) = \Phi(x, y)\,, \;\; \forall x \in \mathbf{R}^{n-1}\,, \forall \Lambda \in \Gamma\,,$$

where Γ is a lattice (cnoidal waves). To fix the ideas we assume in the sequel periodic boundary conditions $x \in \mathbf{T}^{n-1}$, i.e., $\Gamma = 2\pi\mathbf{Z}^{n-1}$.

We want to write the evolution problem (A.1) as an infinite-dimensional Hamiltonian system. Note that the profile $\eta(x)$ of the fluid and the value $\xi(x) := \Phi(x, \eta(x))$ of the velocity potential Φ restricted to the free boundary uniquely determine the velocity potential Φ in the whole of S_η, solving the elliptic problem

$$\begin{cases} \Delta\Phi = 0 & \text{in} \quad S_\eta\,, \\ \partial_y \Phi = 0 & \text{at} \quad y = -h\,, \\ \Phi = \xi & \text{at} \quad y = \eta(x)\,, \end{cases} \tag{A.2}$$

with Φ periodic in x.

We claim that (A.1) can be written, in the variables (ξ, η), as the Hamiltonian system

$$\begin{cases} \partial_t \eta = \delta_\xi H\,, \\ \partial_t \xi = -\delta_\eta H\,, \end{cases} \tag{A.3}$$

with Hamiltonian

$$\begin{aligned} H &= \int_{S_\eta} \frac{1}{2}|\nabla\Phi|^2 \, \mathrm{d}x\mathrm{d}y + \int_{S_\eta} gy \, \mathrm{d}x\mathrm{d}y + \sigma \int_{\mathbf{T}^{n-1}} \sqrt{1 + |\partial_x \eta|^2} \, \mathrm{d}x \\ &:= K + U + \sigma A\,. \end{aligned}$$

Then H is the sum of the kinetic energy, the potential energy, and the area surface integral, expressed in the variables (ξ, η), and δ_η, δ_ξ are the L^2-gradients of H. By (A.3), (ξ, η) are canonically conjugated "Darboux coordinates."

The potential energy is

$$U = \int_{S_\eta} gy \, dx dy = \frac{g}{2} \int_{\mathbf{T}^{n-1}} \eta^2(x) \, dx + \text{constant}.$$

Using the divergence theorem, $\Delta\Phi = 0$, the periodicity, and the Neumann condition at the bottom, the kinetic energy can be expressed as

$$K(\xi, \eta) = \frac{1}{2} \int_{S_\eta} |\nabla\Phi|^2 \, dx dy = \frac{1}{2} \int_{\mathbf{T}^{n-1}} \xi \, G(\eta)[\xi] \, dx , \tag{A.4}$$

where

$$N(x) := \frac{(-\partial_x \eta(x), 1)}{\sqrt{1 + |\partial_x \eta(x)|^2}}$$

is the outer normal, and

$$\begin{aligned} G(\eta)[\xi] &:= \nabla\Phi(x, \eta(x)) \cdot N(x) \sqrt{1 + (\partial_x \eta(x))^2} \\ &= (\partial_y \Phi)(x, \eta(x)) - (\partial_x \Phi)(x, \eta(x)) \cdot \partial_x \eta(x) \end{aligned}$$

is the so-called *Dirichlet–Neumann* operator; see [50].

The operator G is linear in ξ, symmetric with respect to the L^2 scalar product, and semipositive definite.

Remark A.1. It is a deep result that the nonlinear map $\eta \mapsto G(\eta)$ is analytic as a function from $\{\eta \in \mathcal{C}^1 \mid \|\eta\|_{\mathcal{C}^1} < R\}$ into $\mathcal{L}(H^1, L^2)$; see, e.g., [49].

In conclusion, the Hamiltonian is

$$H(\xi, \eta) = \frac{1}{2} \int_{\mathbf{T}^{n-1}} \xi \, G(\eta)[\xi] \, dx + \frac{g}{2} \int_{\mathbf{T}^{n-1}} \eta^2 \, dx + \sigma \int_{\mathbf{T}^{n-1}} \sqrt{1 + |\partial_x \eta|^2} \, dx .$$

Let us prove (A.3). By the symmetry of $G(\eta)$, the partial derivative $\delta_\xi H$ with respect to the L^2 scalar product is

$$\delta_\xi H(\xi, \eta) = G(\eta)[\xi] , \tag{A.5}$$

and the fourth equation in (A.1) coincides with the first equation of (A.3):

$$\partial_t \eta = G(\eta)[\xi] = \delta_\xi H(\xi, \eta) .$$

It is also easy to obtain the partial derivatives

$$\delta_\eta U = g\eta , \qquad \delta_\eta A = -\tau(\eta) . \tag{A.6}$$

It remains to compute the partial derivative of K with respect to the shape domain η,

$$(\delta_\eta K, \nu)_{L^2} = \frac{\partial}{\partial \mu}_{|\mu=0} \int_{S_{\eta+\mu\nu}} \frac{|\nabla \Phi_{\eta+\mu\nu}|^2}{2}, \tag{A.7}$$

where $\Phi_{\eta+\mu\nu}$ is the solution of (A.2) in $S_{\eta+\mu\nu}$ with Dirichlet datum ξ on the varied boundary $y = \eta(x) + \mu\nu(x)$.

Differentiating with respect to μ in (A.7), the term

$$\frac{1}{2} \int_{S_{\eta+\mu\nu}} |\nabla \Phi_{\eta+\mu\nu}|^2 \, dx dy = \frac{1}{2} \int_{\mathbf{T}^{n-1}} \Big(\int_{-h}^{\eta(x)+\mu\nu(x)} |\nabla \Phi_{\eta+\mu\nu}|^2 \, dy \Big) \, dx$$

yields

$$(\delta_\eta K, \nu)_{L^2} = \frac{1}{2} \int_{\mathbf{T}^{n-1}} |\nabla \Phi_\eta(x, \eta(x))|^2 \nu(x) \, dx \; + \; \frac{1}{2} \frac{\partial P}{\partial \mu}_{|\mu=0}, \tag{A.8}$$

where, as in (A.4), we can write

$$P := \int_{S_\eta} |\nabla \Phi_{\eta+\mu\nu}|^2 \, dx \, dy = \int_{\mathbf{T}^{n-1}} \xi_\mu \, G(\eta) \left[\xi_\mu\right] \, dx$$

with[2] $\xi_\mu(x) := \Phi_{\eta+\mu\nu}(x, \eta(x))$. To compute

$$\frac{1}{2} \frac{\partial P}{\partial \mu}_{|\mu=0} = \int_{\mathbf{T}^{n-1}} \Big(\frac{\partial}{\partial \mu}_{|\mu=0} \xi_\mu \Big) G(\eta) \, [\xi] \, dx \tag{A.9}$$

it remains to find

$$\frac{\partial}{\partial \mu}_{|\mu=0} \xi_\mu(x) = \partial_{BC} \Phi_\eta(x, \eta(x)) \, [\nu] \, , \tag{A.10}$$

where ∂_{BC} denotes the derivative of the map $f \mapsto \Phi_f$.

Differentiating the identity

$$\Phi_{\eta+\mu\nu}(x, \eta(x) + \mu\nu(x)) = \xi(x) \qquad \forall \mu$$

at $\mu = 0$ yields

$$\partial_{BC} \Phi_\eta(x, \eta(x)) \, [\nu] = -\partial_y \Phi_\eta(x, \eta(x)) \nu(x) \, , \tag{A.11}$$

and (A.8), (A.9), (A.10), (A.11) give

$$\delta_\eta K = \frac{1}{2} |\nabla \Phi(x, \eta(x))|^2 - (\partial_y \Phi)(x, \eta(x)) \, G(\eta) \, [\xi] \, . \tag{A.12}$$

Differentiating $\xi(x,t) = \Phi(x, \eta(x,t), t)$ with respect to time, thanks to the first equation in (A.1) and $\partial_t \eta = G(\eta)[\xi]$, we get

$$\partial_t \xi = \partial_y \Phi \, \partial_t \eta + \partial_t \Phi = \partial_y \Phi \, G(\eta) \, [\xi] - \frac{1}{2} |\nabla \Phi|^2 - g\eta + \sigma\tau = -\delta_\eta H(\xi, \eta)$$

by (A.6) and (A.12), namely the second equation in (A.3).

[2] We can suppose $\nu(x) > 0$, decomposing in positive and negative part $\nu = \nu^+ - \nu^-$. Hence, for $\mu > 0$, $S_\eta \subset S_{\eta+\mu\nu}$.

Remark A.2. We could express the right-hand side of (A.12) through the quantities (ξ, η) only, yielding

$$\delta_\eta K = \frac{1}{2}\frac{1}{(1+|\partial_x \eta|^2)}\Big[\,|\partial_x \xi|^2 - (G(\eta)\,[\xi])^2 - 2\partial_x \eta \cdot \partial_x \xi\; G(\eta)\,[\xi] \\ + |\partial_x \eta|^2 |\partial_x \xi|^2 - (\partial_x \eta \cdot \partial_x \xi)^2\,\Big];$$

see, e.g., [50]. Note that the last term vanishes for $n = 2$.

Appendix B

Critical Point Theory

We find it convenient to collect briefly the basic ideas and techniques in critical point theory that are used throughout the book.

B.1 Preliminaries

Let $\Phi: X \to \mathbf{R}$ be a C^1-functional defined on a real Banach space X with norm $\|\cdot\|$.

Let $D\Phi(x) \in \mathcal{L}(X, \mathbf{R})$ denote the derivative of Φ at the point x, namely the unique linear and continuous operator from X to $\mathbf{R}$ such that

$$|\Phi(x+h) - \Phi(x) - D\Phi(x)[h]| = o(\|h\|) \quad \text{as} \quad \|h\| \to 0 .$$

For any $L \in \mathcal{L}(X, \mathbf{R}) =: X^*$ in the dual space we define the operatorial norm

$$\|L\|_* := \sup_{\|x\| \le 1} |Lx| .$$

Definition B.1. A point $x \in X$ is called a critical point if $D\Phi(x) = 0$. The corresponding value $\Phi(x) \in \mathbf{R}$ is called a critical value.

When X is a Hilbert space with scalar product $(\cdot, \cdot)$, we denote by $\nabla\Phi(x) \in X$ the gradient of Φ at the point x, uniquely defined, via the Riesz theorem, by

$$(\nabla\Phi(x), h) = D\Phi(x)[h] , \quad \forall h \in X .$$

We have $\|\nabla\Phi(x)\| = \|D\Phi(x)\|_*$.

Given a C^1 function $g: X \to \mathbf{R}$, the set

$$M := g^{-1}(0) = \left\{x \in X \mid g(x) = 0\right\}$$

is a C^1-manifold if $Dg(x) \neq 0$, $\forall x \in M$ (by the implicit function theorem), with tangent space at x

$$T_x M := \left\{h \in X \mid Dg(x)[h] = 0\right\} .$$

Definition B.2. The point $x \in M$ is called a critical point of $\Phi\colon M \subset X \to \mathbf{R}$ constrained to M if $D\Phi(x)[h] = 0, \forall h \in T_x M$.

When X is a Hilbert space and $x \in M$ is a critical point of Φ constrained to M, we have

$$\nabla\Phi(x) = \lambda \nabla g(x)$$

for some $\lambda \in \mathbf{R}$, because $\nabla g(x)$ spans the orthogonal space to $T_x M$. This is called the Lagrange multiplier rule.

B.2 Minima

The simplest critical points to look for are the absolute minima and maxima (as we deal with in Chapter 5).

We first recall the following classical result.

Theorem B.3. (Weierstrass) *Let* $\Phi\colon X \to \mathbf{R}$ *be lower semicontinuous, i.e.,* $\forall x$*,*

$$\forall\, x_n \to x \quad \Longrightarrow \quad \liminf_n \Phi(x_n) \geq \Phi(x)\,.$$

Then on any compact set $K \subset X$*,* Φ *is bounded from below and attains an absolute minimum, i.e., there is* $\bar{x} \in K$ *such that* $\Phi(\bar{x}) = \inf_K \Phi$.

Proof. Take a minimizing sequence $x_n \in K$, namely such that

$$\Phi(x_n) \to \inf_{x \in K} \Phi(x)\,.$$

By the compactness of K, up to a subsequence, $x_n \to \bar{x}$ for some $\bar{x} \in K$. By the lower semicontinuity property of Φ,

$$\inf_K \Phi = \lim_n \Phi(x_n) = \liminf_n \Phi(x_n) \geq \Phi(\bar{x})\,,$$

and therefore, $\inf_K \Phi = \Phi(\bar{x})$, i.e., $\inf_K \Phi$ is finite and $\bar{x} \in K$ is an absolute minimum of Φ in K. ∎

When X is infinite-dimensional, bounded sets are not, in general, precompact ($\equiv$ set with compact closure) in the strong topology, and it is convenient to introduce the weak topology on X. In this weaker topology, bounded sets are precompact; see [41]. We write $x_n \rightharpoonup x$ to indicate that x_n converges to x in the weak topology.

The following theorem is the main tool in the so-called direct methods in the calculus of variations.

Theorem B.4. *Let* X *be a reflexive Banach space and let* $\Phi\colon X \to \mathbf{R}$ *be a functional satisfying the following conditions:*

(Coercivity) $\Phi(x_n) \to +\infty$ *for any sequence* $\|x_n\| \to +\infty$.

(Sequential weakly lower semicontinuity)
$\forall x \in X$, $\forall x_n \rightharpoonup x$ *then* $\liminf_n \Phi(x_n) \geq \Phi(x)$.

Then Φ *is bounded from below and attains an absolute minimum on* X.

Proof. Let $x_n \in X$ be a minimizing sequence, namely $\Phi(x_n) \to \inf_X \Phi$. By the coercivity property, the sequence $\|x_n\|$ is bounded.

Since X is reflexive, up to a subsequence, $x_n \rightharpoonup \bar{x}$ (see, e.g., [41], Theorem III.27). By the weakly lower semicontinuity property,

$$\inf_X \Phi = \lim_n \Phi(x_n) = \liminf_n \Phi(x_n) \geq \Phi(\bar{x}),$$

and therefore, $\Phi(\bar{x}) = \inf_X \Phi$. ∎

In several applications it happens that the functional Φ is not bounded below (or above); or that the absolute (or local) minima correspond to trivial solutions of the problem (for example, in Chapter 2 the trivial solution $v = 0$ is a local minimum of the functional Φ_ε defined in (2.31)).

It is therefore necessary to develop techniques to find "saddle" critical points of unbounded functionals.

B.3 The Minimax Idea

Critical values c of a function $\Phi: \mathbf{R}^n \to \mathbf{R}$ should be found where the topological nature of the sublevels

$$A_c := \Phi^{-1}(-\infty, c]$$

changes, according to the guiding principle that if there are no critical points in $\Phi^{-1}([a, b])$, the sublevel A_b can be deformed into A_a along the trajectories of the negative gradient flow

$$\begin{cases} \dot{\eta}(t, x) = -\nabla\Phi(\eta(t, x)), \\ \eta(0, x) = x, \end{cases}$$

where $t \to \Phi(\eta(t, x))$ is a decreasing function, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\eta(t, x)) = -\|\nabla\Phi(\eta(t, x))\|^2.$$

However, even in finite dimensions, saddle critical points need not in general exist unless some compactness property holds, as illustrated by the following elementary function $\Phi: \mathbf{R}^2 \to \mathbf{R}$ in [126],

$$\Phi(x, y) := \mathrm{e}^{-y} - x^2.$$

The sublevels A_c are disconnected sets for $c < 0$, A_c are connected for $c > 0$, but there are no critical points at the level $c = 0$.

The following strong compactness condition has been introduced by Palais and Smale [106], guaranteeing a critical point in many variational problems.

Definition B.5. (Palais–Smale) (i) A sequence $x_n \in X$ is called a Palais–Smale sequence at the level c ($(\mathrm{PS})_c$ for short) if

$$\Phi(x_n) \to c \quad \text{and} \quad \|D\Phi(x_n)\|_{X^*} \to 0\,.$$

(ii) The functional $\Phi: X \to \mathbf{R}$ is said to satisfy the Palais–Smale condition at the level c ($(\mathrm{PS})_c$ for short) if every $(\mathrm{PS})_c$ sequence $x_n \in X$ is precompact in X; namely, x_n possesses a convergent subsequence $x_{n_k} \to \bar{x}$.

Observe that if $\Phi \in C^1(X, \mathbf{R})$ satisfies the $(\mathrm{PS})_c$ condition, any accumulation point $\bar{x}$ of a $(\mathrm{PS})_c$ sequence x_n is a critical point of Φ at the level c, namely $D\Phi(\bar{x}) = 0$ and $\Phi(\bar{x}) = c$.

The minimax principle is highlighted in the following theorem [105].

Theorem B.6. (Minimax) *Consider $\Phi \in C^1(X, \mathbf{R})$, where X is a Hilbert*[1] *space and suppose $\dot{\eta} = -\nabla\Phi(\eta)$ defines a flow $\eta(t, \cdot): X \to X$, $\forall t \geq 0$.*

(i) *There exists a family of subsets $F \subset X$, say $\mathcal{F}$, positively invariant under the flow, i.e., if $F \in \mathcal{F}$ then $\eta(t, F) \in \mathcal{F}$, $\forall t \geq 0$.*
(ii) *The "minimax" level $c := \inf_{F\in\mathcal{F}} \sup_{x\in F} \Phi(x)$ is finite.*
(iii) *Φ satisfies* $(\mathrm{PS})_c$.

Then c is a critical value.

Proof. We shall prove that for every $\mu > 0$ there exists $x \in X$ such that

$$c - \mu \leq \Phi(x) \leq c + \mu \quad \text{and} \quad \|\nabla\Phi(x)\| \leq \mu\,.$$

Then, choosing $\mu_n := 1/n$, we obtain a Palais–Smale sequence x_n at the level c. By the $(\mathrm{PS})_c$ condition, x_n converges to some critical point in X at the level c.

Arguing by contradiction, there exists $\mu > 0$ such that

$$c - \mu \leq \Phi(x) \leq c + \mu \quad \Longrightarrow \quad \|\nabla\Phi(x)\| > \mu\,. \tag{B.1}$$

By the definition of c there exists an $F \in \mathcal{F}$ such that $\sup_{x\in F} \Phi(x) \leq c + \mu$.

We claim that

$$\eta(t^*, F) \in A_{c-\mu} \qquad \text{for} \qquad t^* := \frac{2}{\mu}\,. \tag{B.2}$$

Since, by (i), $F^* := \eta(t^*, F) \in \mathcal{F}$, (B.2) implies the contradiction

$$c := \inf_{F\in\mathcal{F}} \sup_{x\in F} \Phi(x) \leq \sup_{x\in F^*} \Phi(x) \leq c - \mu\,.$$

To prove (B.2), pick any $x \in A_{c+\mu}$. If $\Phi(\eta(t^*, x)) \leq c - \mu$, we have nothing to prove. If $\Phi(\eta(t^*, x)) > c - \mu$, then $c - \mu < \Phi(\eta(t, x)) \leq c + \mu$, $\forall t \in [0, t^*]$, because $t \to \Phi(\eta(t, x))$ is decreasing. Hence, by (B.1), $\|\nabla\Phi(\eta(t, x))\| > \mu$, and

[1] The same statement holds when X is a complete Finsler manifold of class $C^{1,1}$; see, e.g., [126]. In this case, a "rich" topology of X implies existence of critical points.

$$\Phi(\eta(t^*, x)) = \Phi(x) - \int_0^{t^*} \|\nabla\Phi(\eta(t, x))\|^2 \,\mathrm{d}t < c + \mu - t^*\mu^2 = c - \mu \,.$$

This contradiction proves the claim (B.2). ■

Concerning the verification of the Palais–Smale condition, the first step is often to prove that a Palais–Smale sequence x_n is bounded, i.e., that there exists $C > 0$ such that $\|x_n\| \le C, \forall n$.

This condition is sufficient, for example, in the following situation:

Lemma B.7. *Suppose $D\Phi = L + K$, where $L\colon X \to X^*$ is an invertible linear map with bounded inverse, and $K\colon X \to X^*$ is compact (i.e., maps bounded sets in X in precompact sets in X^*). Then any bounded* $(\mathrm{PS})_c$ *sequence $x_n \in X$ is precompact.*

Proof. We have

$$x_n = L^{-1}D\Phi(x_n) - L^{-1}K(x_n)\,. \tag{B.3}$$

Now, $D\Phi(x_n) \to 0$ because x_n is a Palais–Smale sequence. Moreover, since x_n is bounded and $L^{-1}K$ is compact, then, up to a subsequence, $L^{-1}K(x_n) \to \bar{x}$ in X. We deduce by (B.3) that $x_n \to -\bar{x}$. ■

B.4 The Mountain Pass Theorem

There is no general recipe for choosing the minimax family $\mathcal{F}$, the choice having to reflect some topological change in the nature of the sublevels of Φ.

An important minimax result with a broad spectrum of applications is the celebrated mountain pass theorem due to Ambrosetti and Rabinowitz [7].

Theorem B.8. (Mountain Pass) *Suppose $\Phi \in C^1(X, \mathbf{R})$ and*

(*i*) $\Phi(0) = 0$;
(*ii*) $\exists\, \rho, \alpha > 0$ *such that* $\Phi(x) \ge \alpha$ *if* $\|x\| = \rho$;
(*iii*) $\exists\, v \in X$ *with* $\|v\| > \rho$ *such that* $\Phi(v) < 0$.

Define the "mountain pass" value

$$c := \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} \Phi(\gamma(s)) \ge \alpha\,,$$

where Γ is the minimax class

$$\Gamma := \Big\{\gamma \in C([0,1], X) \mid \gamma(0) = 0\,, \gamma(1) = v\Big\}\,.$$

Then there exists a Palais–Smale sequence of Φ at the level c.

As a consequence, if Φ satisfies the $(\mathrm{PS})_c$ *condition, then c is a critical value for Φ: there exists $\bar{x} \ne 0$ such that $D\Phi(\bar{x}) = 0$ and $\Phi(\bar{x}) = c$.*

Geometrical Interpretation: Think of the graph of Φ as a landscape with a low spot at $x = 0$ surrounded by a ring of mountains. Beyond these mountains lies another low spot at the point υ. Then there must be a mountain pass between 0 and υ containing a critical point. From a topological point of view, the sublevel $A_{c-\varepsilon}$ is not arcwise connected.

Proof. For simplicity we shall give the proof in the case[2] that X is a Hilbert space and $\nabla\Phi$ is locally Lipschitz continuous. In Remark B.9 we explain how to deal with the general case.

First note that for any $\gamma \in \Gamma$, the intersection $\gamma([0,1]) \cap S_\rho$ is nonempty, where $S_\rho := \{x \in X \mid \|x\| = \rho\}$. Therefore

$$\max_{s\in[0,1]} \Phi(\gamma(s)) \geq \min_{x\in S_\rho} \Phi(x) \overset{(ii)}{\geq} \alpha > 0$$

and

$$c := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \Phi(\gamma(s)) \geq \alpha > 0\,.$$

We claim the following:

Claim. $\forall 0 < \mu < c/2$, there exists $x \in X$ such that

$$c - \mu \leq \Phi(x) \leq c + \mu \quad \text{and} \quad \|\nabla\Phi(x)\| < 2\mu\,. \tag{B.4}$$

Then, choosing $\mu_n = 1/n$, we obtain a Palais–Smale sequence x_n at the level c.

The Deformation Argument. For $\mu > 0$ let us consider the sets

$$\tilde{N} := \{x \in X \mid |\Phi(x) - c| \leq \mu \text{ and } \|\nabla\Phi(x)\| \geq 2\mu\}\,,$$
$$N := \{x \in X \mid |\Phi(x) - c| < 2\mu \text{ and } \|\nabla\Phi(x)\| > \mu\}$$

($\tilde{N} \subset N$). Therefore $\tilde{N}$ and N^c are closed, disjoints sets and there exists a locally Lipschitz nonnegative function $g\colon X \to [0,1]$ such that

$$g = \begin{cases} 1 & \text{on } \tilde{N}\,,\\ 0 & \text{on } N^c\,,\end{cases}$$

for example

$$g(x) := \frac{d(x,N^c)}{d(x,N^c) + d(x,\tilde{N})}$$

(here $d(x,\cdot)$ denotes the distance function).

Next consider the locally Lipschitz and bounded vector field

[2] These conditions are met in particular in the applications of this book: in Chapters 1 and 2 the space X is a Hilbert space (actually finite-dimensional in Chapter 1) and $\nabla\Phi \in C^1$; see Remark 2.12.

$$X(x) := -g(x)\frac{\nabla\Phi(x)}{\|\nabla\Phi(x)\|} \tag{B.5}$$

(note that $X(x)$ is well defined because if $\|\nabla\Phi(x)\| < \mu$ then $x \in N^c$ and so $g(x) = 0$).

For each $x \in X$, the unique solution of the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\eta(t,x) = X(\eta(t,x))\,, \\ \eta(0,x) = x\,, \end{cases} \tag{B.6}$$

is defined for all $t \in \mathbf{R}$ (since X is bounded) and

(i) $\eta(t,\cdot)$ is a homeomorphism of X.

Furthermore, since $X(x) \equiv 0$ for $x \in N^c$,

(ii) $\eta(t,x) = x, \forall t$, if $|\Phi(x) - c| \geq 2\mu$ or if $\|\nabla\Phi(x)\| \leq \mu$.

Finally,

(iii) $\forall x \in X, \forall t \in \mathbf{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\eta(t,x)) = -g(\eta(t,x))\|\nabla\Phi(\eta(t,x))\| \leq 0\,. \tag{B.7}$$

Conclusion of the mountain pass theorem. We can now prove the claim (B.4). Arguing by contradiction suppose there exists $0 < \mu < c/2$ such that

$$\forall x \in \{c - \mu \leq \Phi(x) \leq c + \mu\} \quad \Rightarrow \quad \|\nabla\Phi(x)\| \geq 2\mu\,. \tag{B.8}$$

Define $\eta(t,\cdot)$ as in the previous step.

By the definition of $c > 0$ there exists a path $\gamma \in \Gamma$ such that

$$\max_{s\in[0,1]} \Phi(\gamma(s)) \leq c + \mu\,.$$

By (ii), since $\Phi(0) = 0$, $\Phi(v) \leq 0$ and $2\mu < c$,

$$\eta(t,0) = 0\,,\ \eta(t,v) = v\,, \quad \forall t\,. \tag{B.9}$$

Furthermore, by (i), $\eta(t,\gamma(\cdot)) \in C([0,1])$ and therefore $\eta(t,\cdot)\circ\gamma \in \Gamma$ (the minimax family Γ is invariant under the deformations η).

But we claim that

$$\Phi(\eta(1,\gamma([0,1])) \leq c - \mu\,, \tag{B.10}$$

implying the contradiction

$$c := \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} \Phi(\gamma(s)) \leq \max_{s\in[0,1]} \Phi(\eta(1,\gamma(s))) \leq c - \mu\,.$$

Let us prove (B.10). Pick a point $x \in \gamma([0,1])$. If $\Phi(\eta(t,x)) \leq c - \mu$ for some $0 \leq t \leq 1$, there is nothing to prove, since $\Phi(\eta(t,x))$ is decreasing. Assume by contradiction that $\Phi(\eta(t,x)) > c - \mu$ for all $t \in [0,1]$. Therefore, by (B.8),

$$c - \mu \le \Phi(\eta(t,x)) \le c + \mu \quad \text{and} \quad \|\nabla\Phi(\eta(t,x))\| \ge 2\mu\,,$$

i.e., $\eta(t,x) \in \tilde{N}$, $\forall t \in [0,1]$. By (B.7), since $g \equiv 1$ on $\tilde{N}$,

$$\begin{aligned}\Phi(\eta(1,x)) &= \Phi(x) - \int_0^1 g(\eta(t,x))\,\|\nabla\Phi(\eta(t,x))\|\mathrm{d}t \\ &\le (c+\mu) - \int_0^1 \|\nabla\Phi(\eta(t,x))\|\mathrm{d}t \le (c+\mu) - 2\mu = c - \mu\,.\end{aligned}$$

This concludes the proof of the theorem. ∎

The following example from [44] shows a function $\Phi\colon \mathbf{R}^2 \to \mathbf{R}$,

$$\Phi(x,y) := x^2 - (x-1)^3 y^2\,,$$

satisfying the assumptions (i)–(iii) of the mountain pass theorem but without critical points except $(0,0)$.

Indeed, the function Φ does not satisfy the Palais–Smale condition at the mountain pass level c, which is easily verified to be $c = 1$. Any Palais–Smale sequence $(x_n, y_n) \in \mathbf{R}^2$ of Φ at the level $c = 1$ diverges to infinity; more precisely, $x_n \to 1$, $|y_n| \to +\infty$; see [75].

Remark B.9. **(Pseudogradient)** To prove Theorem B.8 in the case that X is a Banach space or $\nabla\Phi$ is not locally Lipschitz, we can replace, in order to construct the deformations $\eta(t,\cdot)$, the gradient $\nabla\Phi(x)$ with a locally Lipschitz vector field $P\colon \{D\Phi(x) \ne 0\} \to X$ such that

$$\begin{cases} \|P(x)\| \le 2\|D\Phi(x)\|_*\,, \\ D\Phi(x)\,[P(x)] \ge \|D\Phi(x)\|_*^2\,. \end{cases}$$

Such a P is called a "pseudogradient" vector field. Note that the second inequality implies also $\|P(x)\| \ge \|D\Phi(x)\|_*$.

Any $\Phi \in C^1(X,\mathbf{R})$ admits a locally Lipschitz pseudogradient vector field P; see, e.g., Lemma A.2 in [119].

We now discuss the minor variants of the mountain pass theorem used in Chapter 1.

Theorem B.10. *Suppose* $\Phi \in C^1(B_r, \mathbf{R})$, *where* $B_r := \{\|x\| < r\}$, *satisfies*

(i) $\Phi(0) = 0$;
(ii) $\exists\, 0 < \rho < r$, $\alpha > 0$ *such that* $\Phi(x) \ge \alpha$ *if* $\|x\| = \rho$;
(iii) $\Phi(x) < 0$, $\forall x \in \partial B_r$.

There exists a $(\mathrm{PS})_c$ *sequence of* Φ *at the level* $c := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \Phi(\gamma(s))$, *where*

$$\Gamma := \Big\{\gamma \in C([0,1], \bar{B}_r) \mid \gamma(0) = 0\,,\ \gamma(1) \in \partial B_r\Big\}\,. \tag{B.11}$$

Proof. Exactly the same proof given for Theorem B.8 above applies, the only difference being to show that the minimax class Γ in (B.11) is invariant under the flow $\eta(t,\cdot)$ defined in (B.6), i.e., if $\gamma \in \Gamma$ then $\eta(t,\cdot) \circ \gamma \in \Gamma, \forall t$.

For this, observe that if $0 < 2\mu < c$, then the vector field X defined in (B.5) vanishes on ∂B_r (since $\forall x \in \partial B_r$, $\Phi(x) < 0$, then $x \in N^c$ and so $g(x) = 0$). Hence the Cauchy problem (B.6) defines a family of deformations $\eta \in C([0,1] \times \bar{B}_r, \bar{B}_r)$ such that $\eta(t,\cdot)_{|\partial B_r} = I$, whence the minimax class Γ is invariant. ■

Proof of (1.61). We follow Lemma 1.19 in [116]. Since V is finite-dimensional, $-\nabla\Phi(1,v)$ is a pseudogradient vector field for $\Phi(\lambda,v)$ for all $v \in \partial Q$ and we can define, by interpolation, say, a pseudogradient vector field $P(v)$ for $\Phi(\lambda,v)$ on $\bar{Q}$ coinciding with $-\nabla\Phi(1,v)$ on ∂Q.

To apply the same argument of the mountain pass theorem, Theorem B.8, again the main issue is to prove that the minimax class Γ defined in (1.60) is invariant under the positive flow generated by $X(v) := -g(v)P(v)/\|P(v)\|$.

This is true by Proposition 1.28: if $v \in \partial Q$ satisfies $\Phi(1,v) = c^- < 0$ (case (*ii*)) then $X(v) \equiv 0$ and therefore $\eta(t,v) = v, \forall t$. In particular, $\eta(t,\cdot)_{|T^-} = I$. If $v \in \partial Q$ satisfies $\Phi(1,v) = c^+$ (case (*i*)) then $\Phi(1,\eta(t,v)) \le c^+$, for $t > 0$, because $P(v)$ is a pseudogradient vector field, and so $\eta(t,v) \in Q$. Otherwise, Proposition 1.28(*iii*) shows that $\eta(t,v) \in \partial Q$. ■

For more material on critical point theory we refer to the monographs by Ambrosetti–Malchiodi [5], Mawhin–Willem [91], Rabinowitz [119], Struwe [126].

Appendix C

Free Vibrations of Nonlinear Wave Equations: A Global Result

As a further application of the mountain pass theorem, Theorem B.8, we prove the following theorem, originally due to Rabinowitz [118], following the proof given by Brezis–Coron–Nirenberg [42].

Theorem C.1. *The nonlinear wave equation*

$$\begin{cases} u_{tt} - u_{xx} + f(u) = 0\,, \\ u(t,0) = u(t,\pi) = 0\,, \end{cases}$$

where $f(u) := |u|^{p-2}u$*,* $p > 2$*, possesses a nontrivial weak* 2π*-periodic solution* $u \in L^p$*.*

Proof. Let

$$N := \Big\{ v = \eta(t+x) - \eta(t-x) \mid \eta \in L^1(\mathbf{T}) \text{ and } \int_0^{2\pi} \eta = 0 \Big\}$$

be the L^1-closure of the classical solutions of the linear wave equation (2.8).

We first collect information on the linear problem

$$\Box \psi = w\,, \qquad \text{where} \qquad \Box := \partial_{tt} - \partial_{xx}\,. \tag{C.1}$$

For any w belonging to the weak orthogonal complement

$$N^{\perp} := \Big\{ w \in L^1(\Omega) \mid \int_{\Omega} w\, v \,\mathrm{d}t\,\mathrm{d}x = 0\,,\ \forall v \in N \cap L^{\infty} \Big\}\,,$$

where $\mathbf{\Omega} := \mathbf{T} \times (0,\pi)$, there exists a unique weak solution $\psi := \Box^{-1} w \in N^{\perp} \cap C(\bar{\Omega})$ of (C.1), namely

$$\int_{\Omega} \psi\, \Box\varphi = \int_{\Omega} w\, \varphi\,, \qquad \forall \varphi \in C_0^2(\bar{\Omega})\,.$$

Actually, by the explicit integral representation formula for $\Box^{-1}w$ given in Remark 5.5, we deduce the estimates

$$\|\Box^{-1}w\|_{L^\infty} \le C\|w\|_{L^1} \tag{C.2}$$

and $\forall q > 1$ (applying Hölder's inequality as in Lemma 5.4),

$$\|\Box^{-1}w\|_{C^\alpha} \le C\|w\|_{L^q} \qquad \alpha := 1 - \frac{1}{q}, \tag{C.3}$$

where C^α denotes the space of the α-Hölder continuous functions.

Let $\Phi \in C^1(W, \mathbf{R})$ be the functional

$$\Phi(w) := \frac{1}{p'}\int_\Omega |w|^{p'} + \frac{1}{2}\int_\Omega w\,\Box^{-1}w\,, \quad \forall w \in W\,,$$

defined on the space

$$W := N^\perp \cap L^{p'}\,,$$

where $p' \in (1,2)$ is the conjugated exponent to p, i.e., $1/p + 1/p' = 1$, endowed with its natural norm

$$\|w\|_{L^{p'}} := \Big(\int_\Omega |w|^{p'}\,\mathrm{d}t\,\mathrm{d}x\Big)^{1/p'}.$$

Note that Φ is well defined because $\Box^{-1}w \in L^\infty(\Omega)$ by (C.2).
We have

$$D\Phi(w)\,[h] = \int_\Omega |w|^{p'-2}w\,h + \int_\Omega h\,\Box^{-1}w, \quad \forall h \in W\,, \tag{C.4}$$

whence, if $w \in W$ is a critical point of Φ,

$$\Box^{-1}w + |w|^{p'-2}w = v$$

for some $v \in N \cap L^p$. Defining

$$u := v - \Box^{-1}w = |w|^{p'-2}w\,,$$

we get $w = |u|^{p-2}u$ and, since $v \in N \cap L^p$,

$$\Box u = -w = -|u|^{p-2}u\,,$$

namely u is a solution of $u_{tt} - u_{xx} + |u|^{p-2}u = 0$.
Therefore, Theorem C.1 will be a consequence of the mountain pass theorem and the following claim.

CLAIM: Φ *satisfies the assumptions of Theorem B.8.*

Step 1: Φ *satisfies assumptions (i)–(iii).*

We have $\Phi(0) = 0$ and

$$\begin{aligned}\Phi(w) &\geq \frac{1}{p'}\|w\|_{L^{p'}}^{p'} - C\|w\|_{L^1}\|\Box^{-1}w\|_{L^\infty} \\ &\overset{(C.2)}{\geq} \frac{1}{p'}\|w\|_{L^{p'}}^{p'} - C'\|w\|_{L^1}^2 \geq \frac{1}{p'}\|w\|_{L^{p'}}^{p'} - C''\|w\|_{L^{p'}}^2 ,\end{aligned}$$

whence, since $1 < p' < 2$, assumption (*ii*) of the mountain pass theorem follows. To verify assumption (*iii*), take $\bar{w} \in W$ such that $\int_\Omega \bar{w}\,\Box^{-1}\bar{w} < 0$ (recall that $\Box^{-1}$ possesses negative eigenvalues; see, e.g., (5.4) and (5.8)). For $t \in \mathbf{R}$ large enough, we have

$$\Phi(t\bar{w}) = \frac{t^{p'}}{p'}\|\bar{w}\|_{L^{p'}}^{p'} + \frac{t^2}{2}\int_\Omega \bar{w}\,\Box^{-1}\bar{w} < 0$$

again because $1 < p' < 2$.

Step 2: Φ *satisfies* $(\mathrm{PS})_c$.

Let w_n be a $(\mathrm{PS})_c$ sequence, namely

$$\Phi(w_n) = \frac{1}{p'}\int_\Omega |w_n|^{p'} + \frac{1}{2}\int_\Omega w_n\,\Box^{-1}w_n \to c \tag{C.5}$$

and $\|D\Phi(w_n)\|_{\mathcal{L}(L^{p'},\mathbf{R})} \to 0$.

By (C.4), using the Hahn–Banach extension theorem, and recalling that the dual space of $L^{p'}$ is L^p, there exists $g_n \in L^p$ such that

$$D\Phi(w_n)\,[h] = \int_\Omega \Big(|w_n|^{p'-2}w_n + \Box^{-1}w_n\Big)h = \int_\Omega g_n\,h\,, \quad \forall h \in W\,, \tag{C.6}$$

with

$$\|g_n\|_{L^p} = \|D\Phi(w_n)\|_{\mathcal{L}(L^{p'},\mathbf{R})} \to 0\,.$$

By (C.6),

$$\Box^{-1}w_n + |w_n|^{p'-2}w_n = g_n + v_n \tag{C.7}$$

with $v_n \in N \cap L^p$.

Subtracting (C.5) from

$$\frac{1}{2}D\Phi(w_n)\,[w_n] = \frac{1}{2}\int_\Omega |w_n|^{p'} + \frac{1}{2}\int_\Omega w_n\,\Box^{-1}w_n = \frac{1}{2}\int_\Omega g_n w_n \tag{C.8}$$

yields

$$\frac{1}{2}D\Phi(w_n)\,[w_n] - \Phi(w_n) = \Big(\frac{1}{2} - \frac{1}{p'}\Big)\int_\Omega |w_n|^{p'}\,,$$

whence

$$\begin{aligned}\Big(\frac{1}{2} - \frac{1}{p'}\Big)\|w_n\|_{L^{p'}}^{p'} &\leq \Big|\frac{1}{2}\int_\Omega g_n w_n\Big| + |\Phi(w_n)| \\ &\leq \frac{1}{2}\|g_n\|_{L^p}\|w_n\|_{L^{p'}} + c + 1\,.\end{aligned}$$

It follows that $\|w_n\|_{L^{p'}} \le C$ is bounded, and therefore, up to a subsequence, $w_n \overset{L^{p'}}{\rightharpoonup} w \in W$ weakly.

By the convexity of the function $|t|^{p'}$ we get

$$\begin{aligned} \frac{1}{p'}|w|^{p'} - \frac{1}{p'}|w_n|^{p'} &\ge |w_n|^{p'-2} w_n (w - w_n) \\ &\overset{(C.7)}{=} (g_n + v_n - \Box^{-1} w_n)(w - w_n)\,, \end{aligned}$$

whence

$$\frac{1}{p'}\int_\Omega |w|^{p'} - \frac{1}{p'}\int_\Omega |w_n|^{p'} \ge \int_\Omega (g_n - \Box^{-1} w_n)(w - w_n)\,, \tag{C.9}$$

since $v_n \in N \cap L^p$.

By (C.3) and the Ascoli–Arzelà theorem, the operator

$$\Box^{-1} \colon W := N^\perp \cap L^{p'} \to L^p \tag{C.10}$$

is compact. Hence, since $w_n - w \rightharpoonup 0$ weakly, $g_n - \Box^{-1} w_n$ converges strongly in L^p, and we deduce that the right-hand side in (C.9) tends to 0, whence

$$\limsup_n \|w_n\|_{L_{p'}} \le \|w\|_{L^{p'}}\,. \tag{C.11}$$

Since the space $L^{p'}$ is uniformly convex, (C.11) and $w_n \overset{L^{p'}}{\rightharpoonup} w$ imply that $w_n \overset{L^{p'}}{\to} w$ strongly (see [41], Proposition III.30). ■

Remark C.2. The proof of Theorem C.1 can be generalized for any superlinear nonlinearity $f(u)$ satisfying

$$\lim_{|u|\to+\infty} \frac{f(u)}{u} = +\infty\,, \qquad \frac{1}{2}u\, f(u) - F(u) \ge \alpha |f(u)| - C\,, \ \ \forall u\,,$$

for some constant $C, \alpha > 0$, where $F(u) := \int_0^u f(s)\, \mathrm{d}s$.

Previous results in the simpler case in which the nonlinearity f grows at most linearly had been obtained in [1], [43]; see also the survey [40].

Remark C.3. (**Regularity**) The weak solution u found in Theorem C.1 is actually in $L^\infty(\Omega)$; see [43]. Furthermore, if $f(u)$ is strictly monotone and of class C^∞, then $u \in C^\infty$; see [43], [118].

Remark C.4. Existence of periodic solutions for any rational frequency $\omega := 2\pi/T \in \mathbf{Q}$ can be proved along the same lines. In contrast, if $\omega \notin \mathbf{Q}$, the compactness property (C.10) no longer holds, and the above proof of the Palais–Smale condition fails.

This is the reflection of small "denominators" phenomena discussed in this book: while for rational frequencies the inverse of the D'Alembert operator is compact (see Lemma 5.4 and (C.3)), for a general irrational frequency it is unbounded (see Section 2.6).

Remark C.5. Nonperturbative existence results of periodic solutions with strongly irrational frequencies (namely badly approximable numbers; see Definition D.5) have been proved in [54], [92], [20], for nonlinearities with linear growth. For these frequencies, as in Lemma 2.10, the inverse of the D'Alembert operator turns out to be at least continuous.

Appendix D

Approximation of Irrationals by Rationals

We collect here some elementary results in number theory used in this book. For a complete presentation we refer to [70], [121].

A real number $\xi \in \mathbf{R}$ may be uniquely written as

$$\xi = [\xi] + \{\xi\},$$

where $[\xi] \in \mathbf{N}$ is the integer part of ξ and $\{\xi\} \in [0, 1)$ its fractional part.

Theorem D.1. (Dirichlet) *Let x be an irrational number. Then*

(i) $\forall Q \in \mathbf{N}$ *there exist integers $q, p \in \mathbf{Z}$ with $1 \le q < Q$ such that*

$$|qx - p| \le \frac{1}{Q}. \tag{D.1}$$

(ii) *There exist infinitely many distinct rational numbers p/q such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{D.2}$$

Proof. **(i)** The $Q + 1$ numbers

$$0, \{x\}, \dots, \{(Q-1)x\}, 1 \in [0, 1]$$

are all distinct since x is irrational. Therefore at least two of them stay at a distance less than or equal to $1/Q$; namely there exist integers $0 \le r_2 < r_1 < Q$, s_1, s_2 such that

$$\left| (r_1 x - s_1) - (r_2 x - s_2) \right| \le \frac{1}{Q}.$$

Setting $q := r_1 - r_2$, $p := s_1 - s_2$, we have $1 \le q < Q$ and (D.1) holds.

(ii) Since x is irrational, $qx - p$ is never zero. Hence as $Q \to +\infty$, we obtain infinitely many distinct pairs of relatively prime integers satisfying (D.1) and therefore (D.2). ∎

Corollary D.2. *If* $x \in \mathbf{R} \setminus \mathbf{Q}$ *the set* $\{xn + m \mid (n, m) \in \mathbf{Z}^2\}$ *is dense in* $\mathbf{R}$.

Proof. $\forall Q \in \mathbf{N}$, by Theorem D.1(**i**), the sequence

$$0, \{qx\}, \ \{2qx\}, \ \{3qx\}, \ \ldots$$

forms a $1/Q$-net of $[0, 1]$. ■

Definition D.3. (Diophantine number) An irrational number x is called Diophantine if there exist constants $\gamma, \tau > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{\gamma}{q^\tau}, \qquad \forall q \in \mathbf{N}, p \in \mathbf{Z}.$$

Note that in view of the Dirichlet theorem, the constant τ has to be greater than or equal to 2, and if $\tau = 2$, the constant γ has to satisfy $\gamma \in (0, 1)$.

Theorem D.4. *Almost every* $x \in \mathbf{R}$ *is Diophantine.*

Proof. It is sufficient to prove that almost every $x \in (0, 1)$ is Diophantine. The complementary set of the Diophantine numbers in $(0, 1)$ for some $\gamma > 0$, $\tau > 2$, is contained in

$$\mathcal{C}_\gamma := \bigcup_{p/q \in \mathbf{Q} \cap (0,1)} \left(\frac{p}{q} - \frac{\gamma}{q^\tau}, \frac{p}{q} + \frac{\gamma}{q^\tau} \right),$$

whose Lebesgue measure is bounded by

$$|\mathcal{C}_\gamma| \leq \sum_{q=1}^{\infty} \sum_{p=1}^{q} \frac{2\gamma}{q^\tau} \leq 2\gamma \sum_{q=1}^{\infty} \frac{q}{q^\tau},$$

which is convergent because $\tau > 2$. We deduce that $|\mathcal{C}_\gamma| \to 0$ as $\gamma \to 0$. ■

Definition D.5. (Badly approximable) A Diophantine number with $\tau = 2$ is called "badly approximable."

Existence of badly approximable numbers is ensured, for example, by the following theorem (see also Theorem D.11 below).

Theorem D.6. (Liouville) *Let* x *be an irrational root of a quadratic polynomial* $P(y) := ay^2 + by + c$ *with* $a, b, c \in \mathbf{Z}$, $a \neq 0$ *(quadratic irrational). Then* x *is badly approximable.*

Proof. If $|x - p/q| > 1$ we have nothing to prove. If $|x - p/q| \leq 1$ then

$$\frac{|ap^2 + bpq + cq^2|}{q^2} = \left| P\left(\frac{p}{q}\right) \right| = \left| P\left(\frac{p}{q}\right) - P(x) \right| \leq M \left| \frac{p}{q} - x \right|, \tag{D.3}$$

where $M := \max_{[x-1,x+1]} |P'(y)| > 0$. Now

$$|ap^2 + bpq + cq^2| \geq 1,$$

for $a, b, c \in \mathbf{Z}$; otherwise, $P(p/q) = 0$, but P cannot have rational solutions. We deduce by (D.3) that

$$\left| x - \frac{p}{q} \right| \geq \frac{M^{-1}}{q^2}.$$

In conclusion: x is badly approximable with $\gamma := \min\{M^{-1}, 1\}$. ■

D.1 Continued Fractions

The classical Euclidean division algorithm allows one to write every real number x in the form of a "continued fraction."

For every $x \in \mathbf{R}$ we define

$$a_0 := [x]\ , \quad x_0 := \{x\}$$

(so that $x = a_0 + x_0$), and inductively, for $n \geq 0$, if $x_n \neq 0$,

$$a_{n+1} := \left[\frac{1}{x_n}\right] \geq 1\ , \quad x_{n+1} := \left\{\frac{1}{x_n}\right\}$$

(so that $x_n^{-1} = a_{n+1} + x_{n+1}$), whence

$$x = a_0 + x_0 = a_0 + \frac{1}{a_1 + x_1} = \cdots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + x_n}}}\ .$$

If $x \in \mathbf{Q}$ is a rational number, this Euclidean division algorithm stops after a finite number of steps (when the remainder x_n is equal to 0).

In contrast, if $x \in \mathbf{R} \setminus \mathbf{Q}$ the previous procedure never ceases, and it defines the so-called continued fraction expansion of the number x,

$$[a_0, a_1, a_2, \dots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}\ .$$

The integers a_n are called the "partial quotients" of x.

EXAMPLE The continuous fraction expansion of the famous "golden ratio" number $x := (\sqrt{5} + 1)/2$ is

$$[1, 1, 1, \dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$$

because x solves the equation $x = 1 + x^{-1}$.

Definition D.7. (Convergents) Given an irrational number x, we define the nth convergents to x as

$$\frac{p_n}{q_n} := [a_0, a_1, a_2, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} \in \mathbf{Q}$$

for $n = 0, 1, 2, \dots$.

D.1. Exercise: Prove, by induction, that p_n, q_n are recursively determined by

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}\,, \\ q_n = a_n q_{n-1} + q_{n-2}\,, \end{cases} \qquad \forall n \geq 0\,,$$

defining $p_{-2} := 0$, $p_{-1} := 0$, $q_{-2} := 1$, $q_{-1} := 0$. In particular, $q_{n+1} > q_n > 0$ and $\lim_{n\to+\infty} q_n = +\infty$ (note that for the golden ratio, q_n, p_n are the Fibonacci sequence). Furthermore,

$$p_n\, q_{n+1} - p_{n+1}\, q_n = (-1)^{n+1}\,, \tag{D.4}$$

which implies in particular that p_n and q_n are relatively prime for each n.

By Definition D.7 and the previous exercise, given $x \in \mathbf{R} \setminus \mathbf{Q}$, the sequence of its convergents satisfy

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < x < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}\,,$$

whence

$$\left|\frac{p_n}{q_n} - x\right| < \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| \overset{(D.4)}{=} \frac{1}{q_n\, q_{n+1}} < \frac{1}{q_n^2}\,, \qquad \forall n \geq 0\,, \tag{D.5}$$

and in particular,

$$\frac{p_n}{q_n} \to x \qquad \text{for} \qquad n \to +\infty\,.$$

The following "inverse" result holds (see Theorem 5C in [121]).

Theorem D.8. (Legendre) *Suppose p, q are relatively prime integers with $q > 0$ and*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}\,.$$

Then p/q is a convergent to x.

The importance of the convergents is finally highlighted by the following theorem due to Lagrange, sometimes called the "law of best approximations" (see Theorem 5E in [121]).

Theorem D.9. (Best approximations) *Let x be an irrational number. Then*

$$|xq_0 - p_0| > |xq_1 - p_1| > |xq_2 - p_2| > \cdots$$

and $\forall 1 \leq q \leq q_n$, $(q, p) \neq (q_n, p_n)$, $n \geq 1$,

$$|xq - p| > |xq_n - p_n|\,.$$

We finally recall the following identity used in Lemma 2.8.

Lemma D.10. *(Theorem 5F in [121]). If* $x \in \mathbf{R} \setminus \mathbf{Q}$ *then*

$$\left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2\Big(\big[a_{n+1}, a_{n+2}, \ldots\big] + \dfrac{1}{\big[a_n, a_{n-1}, \ldots, a_1\big]}\Big)},$$

whence

$$\frac{1}{q_n^2(a_{n+1}+2)} \leq \left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2\, a_{n+1}}. \tag{D.6}$$

By Theorem D.8 and (D.6), arguing as in Lemma 2.8, we deduce the following result:

Theorem D.11. *An irrational* x *is badly approximable if and only if the partial quotients* a_n *in its continued fraction expansion are bounded, namely* $|a_n| \leq K(x)$, $\forall n \geq 0$.

Appendix E

The Banach Algebra Property of $X_{\sigma,s}$

Lemma E.1. *For $s > 1/2$ the space $X_{\sigma,s}$ is an algebra; namely, there exists $c > 0$ such that*

$$\|uv\|_{\sigma,s} \le c\|u\|_{\sigma,s}\|v\|_{\sigma,s}\,, \qquad \forall u, v \in X_{\sigma,s}\,.$$

Proof. Let $u, v \in X_{\sigma,s}$, $u = \sum_{m\in\mathbf{Z}} u_m(x)\,\mathrm{e}^{\mathrm{i}mt}$, $v = \sum_{m\in\mathbf{Z}} v_m(x)\,\mathrm{e}^{\mathrm{i}mt}$. The product uv satisfies

$$uv = \sum_l \Big(\sum_k u_{l-k}v_k\Big)\,\mathrm{e}^{\mathrm{i}lt},$$

so its $X_{\sigma,s}$-norm is, if convergent,

$$\|uv\|^2_{\sigma,s} = \sum_l \Big\|\sum_k u_{l-k}v_k\Big\|^2_{H^1}\big(1+l^{2s}\big)\,\mathrm{e}^{2|l|\sigma}\,.$$

Let us define

$$a_{lk} := \left[\frac{\big(1+|l-k|^{2s}\big)\big(1+k^{2s}\big)}{\big(1+l^{2s}\big)}\right]^{1/2}. \tag{E.1}$$

By Hölder's inequality,

$$\begin{aligned}\Big\|\sum_k u_{l-k}v_k\Big\|^2_{H^1}\big(1+l^{2s}\big) \;&\le\; c_l^2\sum_k \|u_{l-k}v_k\|^2_{H^1}a^2_{lk}\big(1+l^{2s}\big) \qquad \text{(E.2)}\\ &\overset{(E.1)}{=}\; c_l^2\sum_k \|u_{l-k}v_k\|^2_{H^1}\big(1+|l-k|^{2s}\big)\big(1+k^{2s}\big)\,,\end{aligned}$$

where

$$c_l^2 := \sum_k \frac{1}{a^2_{lk}}.$$

We claim that

$$c_l^2 \le c^2\,, \quad \forall l \in \mathbf{Z}\,, \tag{E.3}$$

with

$$c^2 := 2^s \sum_{k\in\mathbf{Z}} \frac{1}{1+k^{2s}} < +\infty\,,$$

which is finite if $s > 1/2$. Let us prove (E.3). By convexity, for $s > 1/2$, we have

$$\begin{aligned} 1+l^{2s} &\le 1+\Big(|l-k|+|k|\Big)^{2s} \le 1+2^{2s-1}\Big(|l-k|^{2s}+k^{2s}\Big) \\ &< 2^{2s-1}\Big(1+|l-k|^{2s}+1+k^{2s}\Big) \end{aligned}$$

and then

$$\frac{1}{a_{lk}^2} < 2^{2s-1}\Big(\frac{1}{1+|l-k|^{2s}}+\frac{1}{1+k^{2s}}\Big). \tag{E.4}$$

By the definition of c_l,

$$\begin{aligned} c_l^2 = \sum_k \frac{1}{a_{lk}^2} &\overset{(E.4)}{<} 2^{2s-1}\Big(\sum_k \frac{1}{1+|l-k|^{2s}}+\sum_k \frac{1}{1+k^{2s}}\Big) \\ &= 2^{2s}\sum_{k\in\mathbf{Z}} \frac{1}{1+k^{2s}} =: c^2 < +\infty. \end{aligned}$$

Finally, using that $H^1(0,\pi)$ is an algebra,

$$\begin{aligned} \|uv\|_{\sigma,s}^2 &= \sum_l \Big\| \sum_k u_{l-k}v_k \Big\|_{H^1}^2 \big(1+l^{2s}\big)\, e^{2|l|\sigma} \\ &\overset{(E.2)}{\le} \sum_l c_l^2 \sum_k \|u_{l-k}\|_{H^1}^2 \|v_k\|_{H^1}^2 \,\big(1+|l-k|^{2s}\big)\,\big(1+k^{2s}\big) e^{2|l|\sigma} \\ &\overset{(E.3)}{\le} c^2 \sum_l \sum_k \|u_{l-k}\|_{H^1}^2 \|v_k\|_{H^1}^2 \,\big(1+|l-k|^{2s}\big)\,\big(1+k^{2s}\big)\, e^{2(|l-k|+|k|)\sigma} \\ &= c^2 \sum_k \Big(\sum_l \|u_{l-k}\|_{H^1}^2 \big(1+|l-k|^{2s}\big)\, e^{2|l-k|\sigma}\Big) \|v_k\|_{H^1}^2 \big(1+k^{2s}\big)\, e^{2|k|\sigma} \\ &= c^2 \|u\|_{\sigma,s}^2 \|v\|_{\sigma,s}^2\,, \end{aligned}$$

proving the algebra property of $X_{\sigma,s}$. Note that the algebra constant c is independent of σ. ∎

Solutions

Problems of Chapter 1

1.1 In polar coordinates,

$$\begin{cases} \rho := \sqrt{x_1^2 + x_2^2}\,, \\ \theta := \arctan(x_2/x_1)\,, \end{cases}$$

system (1.5) can be written

$$\begin{cases} \dot{\rho} = \rho^3\,, \\ \dot{\theta} = 1\,. \end{cases}$$

1.2 The Hamiltonian system

$$\begin{cases} \dot{q}_1 = \omega p_1\,, \\ \dot{p}_1 = -\omega q_1 - 2q_1(q_2q_3 + p_2p_3)\,, \\ \dot{q}_2 = p_2 + (p_1^2 + q_1^2)p_3\,, \\ \dot{p}_2 = -q_2 - (p_1^2 + q_1^2)q_3\,, \\ \dot{q}_3 = -p_3 + (p_1^2 + q_1^2)p_2\,, \\ \dot{p}_3 = q_3 - (p_1^2 + q_1^2)q_2\,, \end{cases}$$

possesses the Lyapunov family of periodic solutions

$$q_1 = A\sin(\omega t + \theta)\,, \quad p_1 = A\cos(\omega t + \theta)\,, \; q_2 = p_2 = q_3 = p_3 = 0\,,$$

with $A, \theta \in \mathbf{R}$.

Problems of Chapter 2

2.1 *Hint:* $|(2-\omega)l - j| = |\omega l - (2l - j)|$. If $2l - j \in \mathbf{N}$, then, $|\omega l - (2l - j)| \geq \gamma/l$, since $\omega \in \mathcal{W}_\gamma$ and $2l - j \neq l$ because $l \neq j$.

Problems of Chapter 3

3.1 Hint: Decompose $u = S(t)u + (I - S(t))u$. Using (3.19)–(3.20) estimate

$$|u|_\lambda \le C_{\lambda_1,\lambda_2}\Big(t^{\lambda-\lambda_1}|u|_{\lambda_1} + t^{-(\lambda_2-\lambda)}|u|_{\lambda_2}\Big)$$

for some positive constant C_{λ_1,λ_2}. Finally minimize in t.

Problems of Chapter 5

5.1 Every solution $\upsilon = \eta(t+x) - \eta(t-x)$ of the linear wave equation (2.8) satisfies

$$\begin{aligned} \upsilon(t+\pi, \pi - x) &= \eta(t+\pi+\pi-x) - \eta(t+\pi-(\pi-x)) \\ &= \eta(t+2\pi-x) - \eta(t+x) \\ &= \eta(t-x) - \eta(t+x) = -\upsilon(t,x)\,. \end{aligned}$$

If υ also satisfies the symmetry condition (5.6), then $\upsilon = 0$.

5.2 *Hint:* the subspace of functions in E satisfying the symmetry condition (5.6) is invariant under the Nemitsky operator induced by the nonlinearity. On this subspace, by Exercise 5.1, the D'Alembertian operator is invertible.

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List of Symbols

d_u the differential with respect to u
∇_u the gradient with respect to u
∇_{L^2} the gradient with respect to the L^2 scalar product
∂_t the partial derivative with respect to t
∂_x the partial derivative with respect to x
u_{tt} the second partial derivative of u with respect to t
u_{xx} the second partial derivative of u with respect to x
$\dot{x}$ the time derivative
D_x the derivative with respect to x
$D^2 H$ the Hessian matrix of H
div, the divergence of a vector field
Δ the Laplacian
Δ^2 the bi-Laplacian
$\Box := \partial_{tt} - \partial_{xx}$ the D'Alembert operator
$L_\omega := \omega^2 \partial_{tt} - \partial_{xx}$
$\mathbf{R}$ the real numbers
$\mathbf{R}^n := \mathbf{R} \times \cdots \times \mathbf{R}$, n times
$|x|$ the Euclidean norm of $x \in \mathbf{R}^n$
i the imaginary unit
$\mathbf{Z}$ the integer relative numbers
$\mathbf{Q}$ the rational numbers
$\mathbf{C}$ the complex numbers
$\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ the 1-dimensional torus
$\mathbf{T}^d := \mathbf{T} \times \cdots \times \mathbf{T}$, d times, the d-dimensional torus
$\boldsymbol{\Omega} := \mathbf{T} \times (0, \pi)$
Mat$(n \times n)$ the set of the $n \times n$ matrices with real entries
J the symplectic matrix
sign( ) the signature of a quadratic form
$\mathrm{d}x \wedge \mathrm{d}y$ the symplectic 2 form in the space $\mathbf{R}^{2n} = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n\}$
$\langle v \rangle$ is the linear vector space generated by v
span$\{e_1, e_2\}$ is the linear vector space generated by e_1, e_2

$\oplus$ the direct sum of two subspaces
$\mathrm{Re}(z)$ the real part of the complex number z
$\bar{z}$ complex conjugated of $z \in \mathbf{C}$
$H_0^1(0, \pi)$ the Sobolev space
L^q the Lebesgue space with norm $\| \, \|_{L^q}$
C^k the class of functions with k continuous derivatives
C^∞ the class of infinitely differentiable functions
$H^j(\Omega)$ the Sobolev spaces
$\mathcal{L}(X, Y)$ the space of continuous linear operators between X and Y
$f(x) = O(x)$ as $x \to 0$ means that there exist constants $C > 0$ and $\delta > 0$ such that $|f(x)| \leq C|x|$ for all $|x| \leq \delta$
$f(x) = o(x)$ as $x \to 0$ means that $\lim_{x \to 0} f(x)/x = 0$
$[a_0, a_1, a_2, \ldots]$ the continued fraction expansion
$[x]$ the integer part of a real number x
$\{x\}$ the fractional part of a real number x
$|A|$ the Lebesgue measure of a measurable set $A \subset \mathbf{R}^n$
X^* the dual space of a Banach space X

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